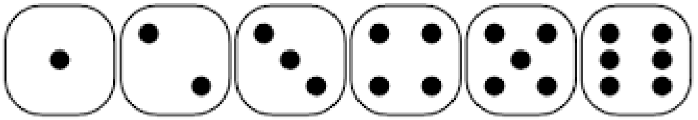
Introduction

- Die / Dice

(Fair Dice)

(Unfair Dice)

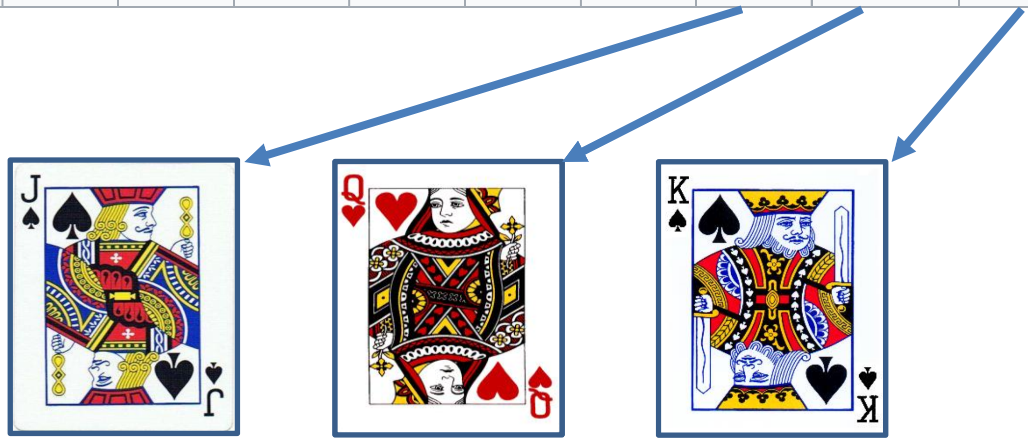


6 sides of a standard die

- Standard 52-card deck

Example set of 52 playing cards; 13 of each suit clubs, diamonds, hearts, and spades

	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs													
Diamonds													
Hearts													
Spades													



Probability

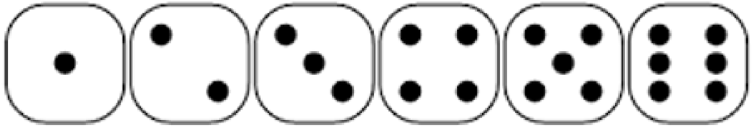
- An experiment is a procedure that yields one of a given set of possible outcomes.

- The set of all possible outcomes of the experiment is called sample space.

- A part of (a subset) the sample space is called an event.

- **Consider a simple action of throwing a single balanced die. There are only six possible outcomes. The set of all possible outcomes associated with this experiment is**

$$\{1, 2, 3, 4, 5, 6\}$$



which is generally referred to as the sample space.

- **The event of observing an odd number can be expressed as the set $\{1, 3, 5\}$.**

– **Definition: A probabilistic experiment, or random experiment, or simply an <u>experiment</u>, is the process by which observation is made.**

– **Example 1: Some examples of experiments are:**
  - **i) Tossing a pair of fair coin**
  - **ii) Throwing a fair die**
  - **iii) Observing the number incoming phone calls to a switchboard during a given hour.**
  - **iv) Determining the amount of dosage that must be given to a patient reacts positively.**

- **Once an experiment has been performed, exactly one of many possible outcomes is observed.**

- **Definition: The sample space associated with an experiment is the set consisting of all possible outcomes of the experiment and is denoted by *S*. The elements of the sample space are referred to as sample points. A discrete sample space is one that contains either a finite or a countable number of distinct sample points.**

- **Example 2: Provide the sample space and identify which of the sample is a discrete sample space:**

i. **Tossing a pair of fair coin**

$$S = \{HH, HT, TH, TT\}$$

ii. **Throwing a fair die**

$$S = \{1, 2, 3, 4, 5, 6\}$$

iii. **Observing the number incoming phone calls to a switchboard during a given hour.**

$$S = \{0, 1, 2, 3, \dots\}$$

iv. **Determining the amount of dosage that must be given to a patient reacts positively.**

$$S = (0, \infty)$$

In the previous example, the sample space for the first two cases have <u>countable</u> and <u>finite</u> sample points

In the third case, it has <u>countably</u> <u>infinite</u> sample points.

And in the last case, it has <u>uncountable</u> set of <u>infinitely</u> many sample points. Thus, the first 3 cases are discrete sample spaces.

- **Example 3: Consider the first experiment in Example 1. For this, the sample space is**

$$S = \{HH, HT, TH, TT\}$$

  Suppose we want to record an outcome of the number of heads observed.

- **Any event associated with an experiment can be categorized as a subset of the sample space.**

- **Definition: An event in a discrete sample space S is any subset of S.**

- **Definition: A <u>simple event</u> is an event that cannot be decomposed. Each simple event corresponds to one and only one sample point. Any event that can be decomposed into more than one simple event is called a <u>compound event</u>.**

- **Definition: Let A be an event connected with an experiment E and let S be the sample space of E. The event B of non-occurrence of A is called the complementary event of A. This means that the subset of B is the complement of A in S.**

Example 4: Consider the experiment of throwing single fair die and recording the number on the top face. The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Express the following events in set forms:

A: the number on the top face is an even number

$$A = \{2, 4, 6\}$$

B: the number on the top face is < 4

$$B = \{1, 2, 3\}$$

C: the number on the top face is prime

$$C = \{2, 3, 5\}$$

D: the number on the top face is 2 or 5

$$D = \{2, 5\}$$

E_1: the number on the top face is 1

E_2: the number on the top face is 2

E_3: the number on the top face is 3

E_4: observe a 4, E_5: observe a 5, E_6: observe a 6

$$E_1 = \{1\}, E_2 = \{2\},$$
$$E_3 = \{3\}, E_4 = \{4\},$$
$$E_5 = \{5\}, E_6 = \{6\}.$$

F: the number on the top face is an odd number

$$F = \{1, 3, 5\} = A'.$$

Note that,

$$C = \{2, 3, 5\} = \{2, 5\} \cup \{3\} = D \cup E_3$$

Hence, C is a compound event.

Similarly, A, B and D are also compound events.

Here $E_1, E_2, \ldots, E_6$ are all simple events.

The event F is the complementary event of A.

– **Example 5: Two six-faced dice are rolled and the numbers rolled are recorded. The sample space is,**

$$S = \left\{ \begin{array}{l} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6), \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6), \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6), \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6), \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6), \end{array} \right\}.$$

i) Let A be the event that the <u>first die shows a 2</u> and the <u>second die shows an odd number</u>. Then,

$$A = \{(2,1),(2,3),(2,5)\}.$$

Note that $|A| = 3$.

ii) **Let B be the event that <u>both dice show the same number</u>. Then,**

$$B = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}.$$

Note that $|B| = 6$.

iii) **Let C be the event that the <u>sum of the numbers rolled is 9</u>. Then,**

$$C = \{(3,6),(4,5),(5,4),(6,3)\}.$$

Note that $|C| = 4$.

QUIZ (Attendance)

iv) **Let *D* be the event that the product of the numbers rolled is 40. Express the event *D* in set form and state the cardinality of *D*.**

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Solution

Now the largest number on both the dice is 6. So, the largest product of the two numbers, one from the first die and the other from the second die is

$$6 \times 6 = 36.$$

Hence, $D = \emptyset$ **and** $|D| = 0$.

- **In an experiment, two or more events are said to be <u>equally likely</u> if, after taking into consideration all relevant evidences, none can be expected in preference to another.**

- **The simple events $\{H\}$ and $\{T\}$ connected with the experiment "tossing a fair coin" are equally likely events.**

- **When a die is rolled, what are the chances that the number rolled is a 2?**

- **Let E denotes the event that the number rolled is 2. Then $E = \{2\}$.**

- **This question can be reformulated: What are the chances that the event E will occur?**

- **In mathematical terms, this question is posed as follows: what is the probability that the number rolled is a 2 or what is the probability that the event E will occur?**

- **The answer of this question depends on whether the die is fair. Suppose that the die is fair. Then when the die is rolled, all the numbers on the die have equal chances of rolling.**

- **Because there are six numbers on the die, we can say that the probability of rolling a 2 is 1/6.**

- **The likelihood of any outcome in a sample space is given by a <u>probability function</u> that assigns to each outcome a real number called the <u>probability</u> of the outcome.**

– **Definition: Let S be a finite sample space of an experiment where each outcome of the experiment is equally likely. Let E be an event in S. Then the probability E to occur, written $Pr[E]$, is given by**

$$Pr[E] = \frac{|E|}{|S|}$$

where $0 \leq Pr[E] \leq 1$.

– **Example 6: Consider the experiment of throwing a single fair die and recording the number on the top face. Because the die is fair, assume that all outcomes of the throws are equally likely.**

Thus, $S = \{1, 2, 3, 4, 5, 6\}$. Consider the following events:

E: the number on the top face is an even number. Then,

$$E = \{2, 4, 6\}.$$

Hence,

$$Pr[E] = \frac{|E|}{|S|} = \frac{3}{6} = \frac{1}{2}.$$

F: the number on the top face is an odd number. Then,

$$E = \{1, 3, 5\}.$$

Hence,

$$Pr[F] = \frac{|F|}{|S|} = \frac{3}{6} = \frac{1}{2}.$$

Note that for the events E and F,

$$E \cap F = \emptyset$$

These two events are called mutually exclusive events.

– **Definition: Let E and F be events in a sample space. The events E and F are said to be mutually exclusive if**

$$E \cap F = \emptyset$$

and

$$Pr[E \cap F] = 0.$$

- **Example 7: Suppose that a card is drawn from a well-shuffled deck of 52 cards. Let S be the sample space. Then, $|S| = 52$.**

  i) Let E be the event that the drawn card is an ace. Find $Pr[E]$.

  Solution

  Since there are 4 aces, then $|E| = 4$. Hence,

$$Pr[E] = \frac{|E|}{|S|} = \frac{4}{52} = \frac{1}{13}.$$

  ii) Let F be the event that the drawn card is a heart or an ace. Find $Pr[F]$.

  Solution

Example set of 52 playing cards; 13 of each suit clubs, diamonds, hearts, and spades

	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs													
Diamonds													
Hearts													
Spades													

There are 13 hearts and 4 aces. However, one of the aces is also a heart. That card is an ace and heart. Thus, $|F| = 13 + 4 - 1 = 16$.

Hence,

$$Pr[F] = \frac{|F|}{|S|} = \frac{16}{52} = \frac{4}{13}.$$

– **Theorem: Let S be a finite sample space such that all outcomes are equally likely and let E and F be events in S. Thus,**

i) $Pr[\emptyset] = 0$

ii) $Pr[S] = 1$

iii) $0 \leq Pr[E] \leq 1$

iv) $Pr[E'] = 1 - Pr[E]$

v) $Pr[E \cup F] = Pr[E] + Pr[F] - Pr[E \cap F]$

vi) $Pr[E - F] = Pr[E \cap F']$

$$= Pr[E] - Pr[E \cap F]$$

- Let S be a finite sample space such that all outcomes are equally likely and let E, F and G be events in S. Thus,
$$Pr[E \cup F \cup G] = Pr[E] + Pr[F] + Pr[G] \\ -Pr[E \cap F] - Pr[F \cap G] - Pr[E \cap G] \\ +Pr[E \cap F \cap G].$$

- Example 8: An experiment consists a forming license plates consisting of three letters followed by three digits, with repetition allowed. Let S be the sample space of the experiment. Then,
$$|S| = 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 1,757,6000.$$

a) Let E be the event that a license plate begins with the letter A. Determine the probability of E.

Solution

To accomplish this first we find the number of elements in E. Now the number of license plates that begin with the letter A is

$$|E| = 1 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 676,000$$

Thus, $|E| = 676,000$. Hence,

$$Pr[E] = \frac{|E|}{|S|} = \frac{676,000}{1,757,6000} = \frac{1}{26}.$$

b) Let F be the event that a license plate begins with a vowel and ends with an even number. Determine the probability of F.

Solution

Because the license plate must begin with a vowel, the first letter has 5 choices. Also, the license plate must end with an even number. So, the last digit can be 0, 2, 4, 6 or 8. So the last digit has only 5 choices. Thus,

$$|F| = 5 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 5 = 1,690,000$$

Hence,

$$Pr[F] = \frac{|F|}{|S|} = \frac{1,690,000}{1,757,6000} = \frac{5}{52}.$$

6.2 Basic Properties of Probabilities

- **In previous examples, we assumed that the outcomes of an experiment are equally likely. However, in real life this may not be true.**

- **Definition: Let $S = \{s_1, s_2, \dots, s_k\}$ be sample space with k outcomes. A probability assignment to the outcome s_i of S is an assignment of a real number p_i, for $i = 1, 2, \dots, k$ such that the following holds:**

  i) $0 \leq p_i \leq 1$, **for** $i = 1, 2, \dots, k$,

  ii) $\sum_{i=1}^{k} p_i = p_1 + p_2 + \cdots + p_k = 1$.

- **If p_i is the probability of the outcome s_i, then we write**

$$Pr[s_i] = p_i$$

for $i = 1, 2, \dots, k$.

- **Example 9: Suppose that a coin is tossed. Then $S = \{H, T\}$. Let $Pr[H] = 0.53$ and $Pr[T] = 0.47$. Now,**

$$0 \le Pr[H] \le 1\ , 0 \le Pr[T] \le 1$$

and

$$Pr[H] + Pr[T] = 0.53 + 0.47 = 1.0.$$

Thus, we have probability assignments $Pr[H] = p_1$ and $Pr[T] = p_2$ where $p_1 + p_2 = 1$.

– **Example 10: Suppose that an experiment has 5 outcomes, $S = \{s_1, s_2, s_3, s_4, s_5\}$ where the probability of these outcomes are given in following table,**

Outcome	s_1	s_2	s_3	s_4	s_5
Probability	0.05	0.35	0.10	0.30	0.20

Now, $0 \leq Pr[s_i] \leq 1$ for all i and

$$\sum_{i=1}^{5} Pr[s_i] = 1.0$$

Thus, we have probability assignment.

– **Definition: Let $S = \{s_1, s_2, \dots, s_k\}$ be sample space of an experiment such that $Pr[s_i] = p_i$ for $i = 1, 2, \dots, k$ is a probability assignment. Let $E = \{y_1, \dots, y_t\}$ be an event in S. Then, $Pr[E]$ is defined by**

$$Pr[E] = Pr[y_1] + \cdots + Pr[y_t].$$

– **Example 11: Suppose that the die is rolled such that**

$$Pr[1] = 0.10, Pr[2] = 0.15, Pr[3] = 0.20,$$
$$Pr[4] = 0.10, Pr[5] = 0.25, Pr[6] = 0.20.$$

Note that this is a probability assignment since $\sum_{i=1}^{6} Pr[i] = 1$. Find $Pr[E]$.

Solution

Let $E = \{1, 3, 6\}$. Then,

$$
\begin{aligned}
Pr[E] &= Pr[1] + Pr[3] + Pr[6] \\
&= 0.10 + 0.20 + 0.20 \\
&= 0.50.
\end{aligned}
$$

- **Definition: Let E be an event such that $Pr[E'] \neq 0$. Then the odds in favour of the event E are defined to be**

$$\frac{Pr[E]}{Pr[E']}.$$

- The odds in favour - the ratio of the number of ways that an outcome can occur compared to how many ways it cannot occur.

- Example 12: Suppose that the probability of winning the next basketball game is 0.48. Let E be the event that the next basketball game will be won. Find the odds in favour of E.

  Solution

Given that,

$$Pr[E] = 0.48.$$

This implies that

$$Pr[E'] = 1 - Pr[E] = 0.52.$$

Thus, the odds in favour of E are

$$\frac{Pr[E]}{Pr[E']} = \frac{0.48}{0.52} = \frac{48}{52} = \frac{12}{13}.$$

Hence, the odds in favour of winning the next basketball game are 12 to 13.

– **Example 13: Suppose that the odds in favour of an event E are 4 to 9. Find the probability of E.**

Solution

Given that, $\frac{Pr[E]}{Pr[E']} = \frac{4}{9}$. Thus,

$$Pr[E] = \frac{4}{9} Pr[E']$$

$$Pr[E] = \frac{4}{9}[1 - Pr[E]]$$

$$Pr[E] = \frac{4}{9} - \frac{4}{9} Pr[E]$$

$$Pr[E] + \frac{4}{9} Pr[E] = \frac{4}{9}$$

$$\left(1 + \frac{4}{9}\right) Pr[E] = \frac{4}{9}$$

$$\frac{13}{9} Pr[E] = \frac{4}{9}$$

$$Pr[E] = \frac{4}{9} \times \frac{9}{13} = \frac{4}{13}.$$

– **Theorem: Let E be an event such that the odds in favour of E are m to n, i.e.,**

$$\frac{Pr[E]}{Pr[E']} = \frac{m}{n}, n \neq 0$$

Then,

$$Pr[E] = \frac{m}{m+n} \text{ and } Pr[E'] = \frac{n}{m+n}.$$

– **Example 14: The odds in favour of winning a bingo game are 5 to 17. Let E be the event of winning the bingo game. Then,**

$$\frac{Pr[E]}{Pr[E']} = \frac{5}{17}.$$

Here $m = 5$ and $n = 17$.

Thus, the probability of winning the bingo game is

$$Pr[E] = \frac{5}{5+17} = \frac{5}{22}$$

and the probability of losing the bingo game is

$$Pr[E'] = \frac{n}{m+n} = \frac{17}{5+17} = \frac{17}{22}.$$

6.3 Conditional Probabilities

– **Conditional probability is the probability of one event occurring with some relationship to one or more other events. For example:**

  a) **Event *A* is that it is raining outside, and it has a 0.3 (30%) chance of raining today.**

  b) **Event *B* is that you will need to go outside, and that has a probability of 0.5 (50%).**

A conditional probability would look at these two events in relationship with one another, such as the probability that it is both raining and you will need to go outside.

– **Definition: Let E and F be two events in a sample space. Then the <u>conditional probability of E</u>, when it is given that the event F has occurred, i.e., $Pr[F] \neq 0$, written $Pr[E|F]$, is given by**

$$Pr[E|F] = \frac{Pr[E \cap F]}{Pr[F]}.$$

– **Example 15: Let E and F be events in a sample space.**

i) **Suppose that $Pr[E] = 0.6, Pr[F] = 0.7$ and $Pr[E \cap F] = 0.3$. Determine $Pr[E|F]$.**

Solution

$$Pr[E|F] = \frac{Pr[E \cap F]}{Pr[F]} = \frac{0.3}{0.7} = \frac{3}{7}.$$

ii) **Suppose that $Pr[E] = 0.65, Pr[F] = 0.75$ and $Pr[E \cup F] = 0.9$. Determine $Pr[E|F]$.**

Solution

Note that,

$$Pr[E \cup F] = Pr[E] + Pr[F] - Pr[E \cap F]$$

$$0.9 = 0.65 + 0.75 - Pr[E \cap F]$$

$$0.9 = 1.40 - Pr[E \cap F]$$

$$0.9 - 1.40 = -Pr[E \cap F]$$

$$-0.50 = -Pr[E \cap F]$$

$$Pr[E \cap F] = 0.5$$

Hence,

$$Pr[E|F] = \frac{Pr[E \cap F]}{Pr[F]} = \frac{0.5}{0.75} = \frac{2}{3}.$$

– **Theorem: Let E and F be events of a sample space S such that $Pr[F] \neq 0$. Then,**

$$Pr[E|F] + Pr[E'|F] = 1.$$

– **Example 16: An urn contains 5 red, 6 brown and 3 white marbles. Two marbles without replacements are drawn. What is the probability that the second marble is brown, given that the first marble is white?**

Solution

Given that, the event where "the first marble is white" has occurred. Thus, F be the event that the 1st marble is white and E be the event that the 2nd marble is brown.

Note that,

$$Pr[E|F] = \frac{Pr[E \cap F]}{Pr[F]}.$$

The total number of marbles is 14. Because two marbles, without replacement, are selected,

$$|S| = 14 \cdot (14 - 1) = 14 \cdot 13.$$

Now, F is the event that the first marble is white. So, the second marble can be any marble. Thus,

$$|F| = 3 \cdot (14 - 1) = 3 \cdot 13$$

Hence,

$$Pr[F] = \frac{|F|}{|S|} = \frac{3 \cdot 13}{14 \cdot 13} = \frac{3}{14}.$$

Next, $E \cap F$ is the event that the 1st marble is white and the 2nd marble is brown. Thus,

$$|E \cap F| = 3 \cdot 6$$

Hence,

$$Pr[E \cap F] = \frac{|E \cap F|}{|S|} = \frac{3 \cdot 6}{14 \cdot 13}$$

Now, we have the values for $Pr[E \cap F]$ and $Pr[F]$. We can proceed to compute $Pr[E|F]$. Therefore,

$$Pr[E|F] = \frac{Pr[E \cap F]}{Pr[F]} = \frac{\frac{3 \cdot 6}{14 \cdot 13}}{\frac{3}{14}} = \frac{6}{13}.$$

– **Theorem: Let E and F be two events of an experiment.**

a) **Suppose that $Pr[F] \neq 0$. Then,**

$$Pr[E \cap F] = Pr[F] \cdot Pr[E|F]$$

since

$$Pr[E|F] = \frac{Pr[E \cap F]}{Pr[F]}.$$

b) **Suppose that $Pr[E] \neq 0$. Then,**

$$Pr[E \cap F] = Pr[E] \cdot Pr[F|E]$$

since

$$Pr[F|E] = \frac{Pr[E \cap F]}{Pr[E]}.$$

– **Definition: Let E and F be two events of an experiment. Then E and F are called independent if**

$$Pr[E \cap F] = Pr[E] \cdot Pr[F].$$

– **Theorem: Let E and F be two independent events of an experiment.**

a) If $Pr[F] \neq 0$, then,

$$Pr[E|F] = Pr[E]$$

i.e., the outcome E does not depend on the outcome of F.

Proof

Since E and F are two independent events, then

$$Pr[E|F] = \frac{Pr[E \cap F]}{Pr[F]} = \frac{Pr[E] \cdot Pr[F]}{Pr[F]} = Pr[E].$$

b) If $Pr[E] \neq 0$, then,

$$Pr[F|E] = Pr[F]$$

i.e., the outcome F independent from the outcome of E.

– **Exercise: To exercise regularly, a survey of 100 students shows that 40% play basketball, 70% jog and 82% play either basketball or jog. Let B be the event that a randomly selected student plays basketball and J be the event that a randomly selected student jogs. Show that the events B and J are independent.**

Hints

The events B and J are independent if

$$Pr[B \cap J] = Pr[B] \cdot Pr[J].$$

The values for $Pr[B]$ and $Pr[J]$ are given. Now, find the value of $Pr[B \cap J]$ using all the given information.

Solution

Given that,

$$Pr[B] = 0.40, Pr[J] = 0.70 \text{ and } Pr[B \cup J] = 0.82$$

Now,

$$Pr[B \cup J] = Pr[B] + Pr[J] - Pr[B \cap J]$$

$$0.82 = 0.40 + 0.70 - Pr[B \cap J]$$

$$0.82 = 1.10 - Pr[B \cap J]$$

$$0.82 - 1.10 = Pr[B \cap J]$$

$$-0.28 = -Pr[B \cap J]$$

$$Pr[B \cap J] = 0.28$$

Since $Pr[B] \cdot Pr[J] = 0.28 = Pr[B \cap J]$, **therefore the events** B **and** J **are independent.**

REFERENCES

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[3] https://www.statisticshowto.com