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Symmetric Bernstein Polynomial Approach for the System of Volterra Integral Equations on Arbitrary Interval and Its Convergence Analysis

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Abstract: In this paper, a new numerical technique is introduced to find the solution of the system of Volterra integral equations based on symmetric Bernstein polynomials. The use of Bernstein polynomials to find the numerical solutions of differential and integral equations increased due to its fast convergence. Here, the numerical solution of the system of Volterra integral equations on any finite interval $[m, n]$ is obtained by replacing the unknown functions with the generalized Bernstein basis functions. The proposed technique converts the given system of equations into the system of algebraic equations which can be solved by using any standard rule. Further, Hyers–Ulam stability criteria are used to check the stability of the given technique. The comparison between exact and numerical solution for the distinct nodes is demonstrated to show its fast convergence.

Keywords: Bernstein basis function; discretization; convergence analysis; physical model; stability analysis

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1. Introduction

Volterra integral equations (VIEs) appear in many physical and mechanical engineering models. The use of VIEs has widely increased in dynamics of bridges, railways, and runways, as well as missiles and other structures. In this section, we will discuss the recent development for the numerical solution of the system of VIEs by different researchers using different techniques. Akbar et al. [1] found an analytical solution using an optimal homotopy asymptotic method (OHAM) for the system of VIEs. The proposed technique uses auxiliary functions containing auxiliary convergence control constants. Mutaz [2] proposed the Biorthogonal wavelet-based method to find the numerical results of the system of VIEs. Gu [3] presented the piecewise collocation approach for the system of VIEs. The main purpose of this research was to compare the convergence rate with the global spectrum collocation method. Farshid [4] presents a numerical computational solution via rationalized Haar functions of the linear VIEs system by converting the linear system to the system of algebraic equations. Niazi et al. [5] proposed a collocation technique for the numerical solution of the systems of linear VIEs with variable coefficients. The basic idea was to use Bessel polynomial and collocation points to transform the given equations into matrix equation. Wei and Zhong [6] introduced a new reproducing kernel method for the solution of the system of linear VIEs. The main purpose of this work was to use reproducing kernel theory to obtain the numerical solution for linear VIEs of the second kind. Navid and Emran [7] proposed a spectral method for solving the systems of VIEs. This method converts the linear system of VIEs into the matrix equation. In nonlinear cases, these equations are solved by any standard rule after applying the proposed method. Masouri et al. [8] worked on the VIEs of the first kind by introducing an expansion iterative method. The main task of this work was to find the numericlal solution of the given problem without converting into algebraic equations. Sorkun and Salih [9] proposed the numerical solution of linear VIEs systems with variable coefficients. The main idea of this research was to use Taylor series to transform the integral system into matrix equations. To solve the second kind system of VIEs, Berenguer et al. [10] used the Biorthogonal systems. The main task was to use the Biorthogonal system in Banach spaces in which the problem of approximating the solution is turned into a numerical method. Armand and Gouyandeh [11] tried to find the approximate results for the system of first kind VIEs by converting the VIEs to Volterra integro-differential equations. The comparison of Variational iteration method with the modified variational iteration method was presented. Katani and Shahmorad [12] proposed the block-by-block method for the systems of nonlinear VIEs in which order of convergence was easily achieved. Hossein and Biazar [13] introduced a new numerical approach for the solution of the systems of VIEs in which a modified homotopy perturbation method was presented for the second kind VIEs. Yang et al. [14] described the reproducing kernel approach with variable coefficients to solve the system of the linear VIEs. Sidorov [15] presented the solvability of systems with piecewise continuous kernels of VIEs of the first Kind. Jafarian et al. [16] presents the artificial neural networks-based modeling for the VIEs. Bownds and Cushing [17] gives some stability theorems for the systems of Volterra integral equations. Biazar and Ebrahimi [18] uses the Chebyshev wavelets approach for nonlinear systems of VIEs. Biazar and Eslami [19] work on the system of VIEs of the second kind using modified homotopy perturbation method of the second kind. In recent years, many methods have been proposed by the different authors to find the numerical solution of some physical models of integral equations [20–22].

In view of the above illustrated literature, no attempt has been observed for the approximate solution of the system of VIEs on arbitrary interval $[m, n]$. So a new approach is presented using Bernstein basis functions to solve the system of VIEs.

The following is the general form of the system of VIEs of the first and second kind, respectively.

$$f_p(h) = \sum_{q=1}^{s} \lambda_{p,q} \int_m^h K_{p,q}(h,k) v_q(k) dk, \tag{1}$$

$$\sum_{q=1}^{s} g_{p,q}(h) v_p(h) = f_p(h) + \sum_{q=1}^{s} \lambda_{p,q} \int_m^h K_{p,q}(h,k) v_q(k) dk, \tag{2}$$

where $p = 1, \ldots, s$, $\lambda_{p,q}$ are the constant parameters, $K_{p,q}(h,k)$ and $f_p(h)$ are predetermined real-valued functions, and $v_q(h) = \{v_1(h), \ldots, v_s(h)\}$ is the vector solution to be determined.

Any real valued function v_q defined on the arbitrary interval $[m, n]$ can be written in terms of Bernstein's polynomial as

$$B_r(v_q; h) \approx \sum_{l=0}^{r} v_{q,l} B_{l,r}(h), \tag{3}$$

where

$$B_{l,r}(h) = {}^{r}C_l \frac{(h-m)^l (n-h)^{r-l}}{(n-m)^r}, \qquad l = 0, \ldots, r. \tag{4}$$

where ${}^{r}C_l = \frac{n!}{r!(n-r)!}$, r is the degree of Bernstein polynomial and $B_{l,r}(h)$ is the Bernstein basis function and $v_{q,l}(h) = \{v_{q,1}(h), \ldots, v_{q,s}(h)\}$ is the vector solution to be determined. There are $r+1$, rth degree Bernstein polynomials and belong to the space of polynomial $\{1, h, h^2, \ldots, h^r\}$.

2. The Proposed Approach

In this section, the description of the given technique using Bernstein basis approximation is given, and is used to find the approximate results of VIEs of the first and second kind.

Bernstein's Approximation for the VIEs of the First Kind

To find the approximate solution of the system of VIEs of the first kind given in Equation (1), the unknown function $v_q(k)$ is replaced with the Bernstein basis function of degree r defined in Equation (3). Then, for $p = 0, \dots, s$, Equation (1) becomes,

$$f_p(h) = \sum_{q=1}^{s} \sum_{l=0}^{r} \lambda_{p,q} \int_m^h K_{p,q}(h,k) v_{q,l} {}^rC_l \frac{(k-m)^l (n-k)^{r-l}}{(n-m)^r} dk. \tag{5}$$

To calculate the values of $v_{q,l}$, Equation (5) can be written as a system of linear equations by replacing h with $h_z = m + \frac{(n-m)z}{r} + \epsilon$, $z = 0, \dots, r-1$ and $h_r = r - \epsilon$, where $0 < \epsilon < 1$.

The following linear algebraic system is obtained

$$\sum_{q=1}^{s} \sum_{l=0}^{r} y_{p,q}^{h,k} v_{q,l} = f_p(h_z), \qquad p = 1, \dots, s, \tag{6}$$

where

$$y_{p,q}^{h,k} = {}^rC_l \frac{1}{(n-m)^r} \lambda_{p,q} \int_m^h K_{p,q}(h_m, k)(k-m)^l (n-k)^{r-l} dk.$$

Now, the system Equation (6) can be written in the matrix form as

$$PY = Q, \tag{7}$$

where

$$P = \begin{bmatrix} P'_{1,1} & P'_{1,2} & \dots & P'_{1,s} \\ P'_{2,1} & P'_{2,2} & \dots & P'_{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ P'_{s,1} & P'_{s,2} & \dots & P'_{s,s} \end{bmatrix},$$

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_s \end{bmatrix}, Q = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_s \end{bmatrix},$$

with the block matrices

$$P'_{p,q} = \begin{bmatrix} y_{p,q}^{0,0} & \dots & y_{p,q}^{0,r} \\ y_{p,q}^{1,0} & \dots & y_{p,q}^{1,r} \\ \vdots & \ddots & \vdots \\ y_{p,q}^{r,0} & \dots & y_{p,q}^{r,r} \end{bmatrix},$$

$$y_q = \begin{bmatrix} y_{q,0} \\ y_{q,1} \\ \vdots \\ y_{q,r} \end{bmatrix}, Q_j = \begin{bmatrix} f_p(h_0) \\ f_p(h_1) \\ \vdots \\ f_p(h_z) \end{bmatrix}.$$

Therefore, Equation (7) can be solved by using any standard rule to obtain the unknowns $v_{q,l}$. Then, by using these unknowns in Equation (3), we have $B_r(v_q, h)$, which is the required approximate solution.

The discretization of VIEs of the second kind can be obtained by following the same steps starting from Equation (2).

In the next section, we will check the stability analysis and convergence of the proposed method.

3. Hyers–Ulam Stability and Convergence Analysis

Some important results are given in this section about convergence and stability analysis of the proposed technique. Firstly, some results are given, which will be used to prove our main result.

Theorem 1. *The sequence* $\{B_r(v_q; h)\}$ *where* $r = 1, 2, \ldots$ *converges to* $v_q(h)$*, where* $v_q(h)$ *is any continuous function, and*

$$\|B_r(v_q; h) - v_q(h)\| < \epsilon, \qquad h \in [0, 1]. \tag{8}$$

where $B_r(v_q; h)$ *is given in Equation (3).*

Proof. See [23]. □

In [23], it is given in the first chapter that r must satisfy the condition $r > \frac{Y}{\delta^2 \epsilon}$ *where* $Y = \|v\|$*, where* $v_q(h)$ *is uniformly continuous on* $[0, 1]$*. Now, we will find the condition on r for arbitrary interval* $[m, n]$*.*

Theorem 2. *If* $\|v_q''(h)\|$, $q = 1, \ldots, s$ *are continuous on the interval* $[m, n]$ *and* $\epsilon > 0$*, then* $\exists$ *a Bernstein polynomial approximation* $\|B_r(v_q; h)\|$ *to the function* $v_q(h)$*, such that* $\|B_r(v_q; h) - v_q(h)\| < \epsilon$*. Suppose* $Y = \|v_q\|$*, then r will satisfy* $r > \frac{Y}{\delta^2 \epsilon}(m + n)(2 - m - n)$*. Hence,* $v_q(h)$*,* $q = 1, \ldots, s$*, is uniformly continuous on the interval* $[m, n]$*.*

Proof. See [24]. □

Theorem 3. *If* $v_q(h)$*,* $q = 1, \ldots, s$ *is differentiable to some neighborhood of h and is bounded on* $[m, n]$*. Suppose* $v_q(h)$ *has the second derivative* $v_q''(h)$ *for some* $h \in [m, n]$*, then*

$$\lim_{r \to \infty} r(B_r(v_q; h) - v_q(h)) = \frac{(h - m)(n - h)}{2} v_q''(h). \tag{9}$$

If we take $\|.\|$ *as the maximum norm, then the error bound is*

$$\|B_r(v_q; h) - v_q(h)\| \leq \frac{(n - m)^2}{8r} \sum_{q=1}^{s} \|v_q''(h)\|. \tag{10}$$

Proof. See [25]. □

Theorem 4. *Let $K_{p,q}(h,k)$ be continuous on the interval $[m,n]$. The solution of Equation (1) belongs to $(C^{\gamma} \cap L^2)(m \leq p \leq n)$ for some $\gamma > 2$. Let Y be the matrix defined in Equation (7). If $|Y| \neq 0$, then*

$$\begin{aligned} &\sup_{h_z \in [m,n]} |v_q(h_z) - B_r(v_{q,r}; h_z)| \\ &\leq \frac{(n-m)^2}{8r} \sum_{q=1}^{s} \Big[\|\eta\| Y^{-1} \| \|v_q''\| + \|v_{q,r}''\| \Big], \end{aligned} \tag{11}$$

where $h_z = m + \frac{(n-m)z}{r}$, $z = 0, \ldots, r$, $\eta = \sup\limits_{p,q \in [m,n]} |\lambda_{p,q} K_{p,q}(h,k)|$ and $v_q(h)$, $q = 1, \ldots, s$, $p = 1, \ldots, s$, is the exact solution and $B_r(v_{q,r}; h)$ is the required approximate solution.

Proof. Consider

$$\begin{aligned} \sup_{h_z \in [m,n]} |v_q(h_z) - B_r(v_{q,r}; h_z)| &= \sup_{h_z \in [m,n]} |v_q(h_z) - v_{q,r}(h_z) + v_{q,r}(h_z) - B_r(v_{q,r}; h_z)| \\ &\leq \sup_{h_z \in [m,n]} |v_q(h_z) - v_{q,r}(h_z)| + \sup_{h_z \in [m,n]} |v_{q,r}(h_z) - B_r(v_{q,r}; h_z)|. \end{aligned} \tag{12}$$

By Theorem 3

$$\sup_{h \in [m,n]} |v_{q,r}(h) - B_r(v_{q,r}; h)| \leq \frac{(n-m)^2}{8r} \sum_{q=1}^{s} \|u_{q,r}''\|. \tag{13}$$

Now, we will use Bernstein approximation to find the bound for $\sup\limits_{h_z \in [m,n]} |v_q(h_z) - v_{q,r}(h_z)|$.

Hence, Equation (1) becomes

$$f_p(h) = \sum_{q=1}^{s} \lambda_{p,q} \int_m^h K_{p,q}(h,k) B_r(v_q; k) dk. \qquad p = 1, \ldots, s, \tag{14}$$

If $v_q(h)$ is replaced with $v_{q,r}(h)$, then $f_p(h)$ becomes $\hat{f}_p(h)$. Consequently, the new equation is

$$\hat{f}_p(h) = \sum_{q=1}^{s} \lambda_{p,q} \int_m^h K_{p,q}(h,k) B_r(v_{q,r}; k) dk. \tag{15}$$

Now, if h is to be replaced by $h_z = m + \frac{(n-m)z}{r}$, we have

$$f_p(h_z) = v_q(h_z) Y, \tag{16}$$

and

$$\hat{f}_p(h_z) = v_{q,r}(h_z) Y, \tag{17}$$

where Y is the matrix defined in (7). So

$$v_q(h_z) - v_{q,r}(h_z) = Y^{-1} \Big[f_p(h_z) - \hat{f}_p(h_z) \Big]. \tag{18}$$

Taking the norm, we have

$$\sup_{h_z \in [m,n]} |v_q(h_z) - v_{q,r}(h_z)| \leq \|Y^{-1}\| \max |f_p(h_z) - \hat{f}_p(h_z)|. \tag{19}$$

Next, we have to get a bound for $\max|f_p(h_z) - \hat{f}_p(h_z)|$. Take

$$f_p(h) = \sum_{q=1}^{s} \lambda_{p,q} \int_m^h K_{p,q}(h,k) v_q(k) dk \tag{20}$$

and

$$\hat{f}_p(h) = \sum_{q=1}^{s} \lambda_{p,q} \int_m^h K_{p,q}(h,k) B_r(v_{q,r};k) dk, \tag{21}$$

so that

$$f_p(h) - \hat{f}_p(h) = \sum_{q=1}^{s} \lambda_{p,q} \int_m^h K_{p,q}(h,k)(v_q(k) - B_r(v_{q,r};k)) dk. \tag{22}$$

Taking supremum, we get

$$\sup|f_p(h) - \hat{f}_p(h)| = \sup\left|\sum_{q=1}^{s} \lambda_{p,q} \int_m^h K_{p,q}(h,k)\big(v_q(k) - B_r(v_{q,r};k)\big) dk\right|, \tag{23}$$

which implies

$$\sup|f_p(h) - \hat{f}_p(h)| \leq \sum_{q=1}^{s} \sup|\lambda_{p,q} K_{p,q}(h,k)(v_q(k) - B_r(v_{q,r};k))|. \tag{24}$$

Let $\sup\limits_{h,k\in[m,n]} |\lambda_{p,q} K_{p,q}(h,k)| = \eta$, then

$$\sup|f_p(h) - \hat{f}_p(h)| \leq \sum_{q=1}^{s} \eta \frac{(n-m)^2}{8r} \|v_q''\|,$$

using this result in inequality Equation (19), we get

$$\sup_{h_z\in[m,n]} |v_q(h_z) - v_{q,r}(h_z)| \leq \eta \frac{(n-m)^2}{8r} \|Y^{-1}\| \sum_{q=1}^{s} \|v_q''\|. \tag{25}$$

Hence, Equations (12), (13) and (25) imply

$$\sup_{h_z\in[m,n]} |v_q(h_z) - B_r(v_{q,r};h_z)| \leq \frac{(n-m)^2}{8r} \sum_{q=1}^{s} \Big[\|\eta\|Y^{-1}\|\|v_q''\| + \|v_{q,r}''\|\Big]. \tag{26}$$

which completes the proof. □

The major deficiency of the above result is the presence of $\|Y^{-1}\|$. So we have to find a bound for $\|Y^{-1}\|$ by adding some extra conditions.

Lemma 1. *Let I be the unit matrix of order $s(r+1)$. Consider $\|Y - I\| = \delta < 1$, then*

$$\|Y^{-1}\| \leq \frac{1}{1-\epsilon},$$

with the condition

$$Cond(Y) \leq \frac{\sum_{q=1}^{s} \gamma_{p,q}}{1-\epsilon},$$

where $\gamma_{p,q} = \max_r |\lambda_{p,q} \int_m^h K_{p,q}(h_z,k)dk|$.

Proof. Firstly, we set a bound for $\|Y\|$, where

$$\|Y\| = \max_r \sum_{q=1}^{s} \sum_{l=0}^{r} \left| {}^rC_l \lambda_{p,q} \int_m^h K_{p,q}(h_z,k) \frac{(k-m)^l (n-k)^{r-l}}{(n-m)^r} dk \right|.$$

Bernstein polynomials form the partition of unity, so

$$\|Y\| = \max_r \left| \sum_{q=1}^{s} \lambda_{p,q} \int_m^h K_{p,q}(h_z,k)dk \right|. \tag{27}$$

which implies

$$\|Y\| \leq \max_r \sum_{q=1}^{s} |\lambda_{p,q} \int_m^h K_{p,q}(h_z,k)dk|. \tag{28}$$

This further implies

$$\|Y\| \leq \sum_{q=1}^{s} \gamma_{p,q}, \tag{29}$$

where $\gamma_{p,q} = \max_r |\lambda_{p,q} \int_m^h K_{p,q}(h_z,k)dk|$. Now, we will find the bound of $\|K^{-1}\|$. Suppose

$$K = Y - I,$$

so

$$\|K\| = \|Y - I\| = \delta < 1. \tag{30}$$

Since $Y = I + K$ implies $Y^{-1} = (I+K)^{-1}$, so

$$\|Y^{-1}\| = \|(I+K)^{-1}\|. \tag{31}$$

After applying the Geometric series, we get

$$\|Y^{-1}\| \leq \frac{1}{1-\|K\|} \leq \frac{1}{1-\epsilon}. \tag{32}$$

Hence, the required condition is

$$Cond(Y) = \|Y\| \, \|Y^{-1}\| \leq \frac{\sum_{q=1}^{s} \gamma_{p,q}}{1-\epsilon},$$

which completes the proof. □

The results and their proofs for the system of VIEs of the second kind can be obtained by following the same steps, starting from Equation (2).

In the next section, we will analyze the stability of the given technique by using Hyers–Ulam stability criteria [26]. Before this, some definitions are given which will be helpful to prove our main result.

Definition 1. *(Hölder inequality) Let $c > 1$, $\frac{1}{c} + \frac{1}{d} = 1$, $v \in L^c(E)$, $w \in L^d(E)$ the $vw \in L(E)$ and*

$$\int_E |v(h)w(h)|dh \leq \left(\int_E |v(h)|^c dh\right)^{\frac{1}{c}} \left(\int_E |w(h)|^d dh\right)^{\frac{1}{d}}.$$

Definition 2. *Assume that $M : V \to V$ is a strictly contractive operator and let (X,d) be a complete metric space and $d(Mp, Mq) \leq \theta d(p,q)$ where $\theta \in (0,1)$. Then $(Mp^* = p^*)$, where p^* is the unique fixed point of the mapping M and the sequence $\{M^n p\}$ converges to p^*.*

Theorem 5. *(Stability of VIEs of the second kind) Suppose that $B_r(v_q; h) : [m,n] \to \mathbb{R}$, $f_p(h) \in L^2([m,n])$ and $K_{p,q}(h,k) \in L^2([m,n] \times [m,n])$. If $B_r(v_q; p)$ satisfies the following inequality*

$$\left|\sum_{q=1}^{s} g_{p,q}(h)B_r(v_q;h) - f_p(h) - \sum_{q=1}^{s} \lambda_{p,q} \int_m^h K_{p,q}(h,k)B_r(v_q;k)dk\right| \leq \epsilon \quad (\epsilon \geq 0) \tag{33}$$

where $\left|\sum_{q=1}^{s} \lambda_{p,q} \int_m^h K_{p,q}(h,k)dk\right| \leq \mu < 1$, then there are the solutions v_q satisfying Equation (2), and for $r > \frac{S}{\delta^2 \epsilon}(m+n)(2-m-n)$ and $S = \|v_q\|$, we have $\|B_r(v_q;h) - v_q(h)\| < \epsilon$.

Proof. Let M be an operator defined by:

$$\begin{aligned} &(Mq_n)(h) = f_p(h) + g_p(h) \\ &+ \sum_{m=1}^{n} \lambda_{p,q} \int_m^h K_{p,q}(h,k)v_q(k)dk, \; v_q \in L^2([m,n]). \end{aligned} \tag{34}$$

for $q = 1, \ldots, s$. Now, using Definition 1, we get

$$\begin{aligned} &\int_m^h \left|\int_m^h K_{p,q}(h,k)v_q(k)dk\right|^2 dh \\ &\leq \int_m^h \left[\int_m^h (K_{p,q}(h,k))^2 dk \int_m^h (v_q(k))^2 dk\right] dh \\ &\leq \int_m^h \big(v_q(k)\big)^2 dk \cdot \int_m^h \int_m^h \big(K_{p,q}(h,k)\big)^2 dkdh \leq \infty, \end{aligned}$$

which implies that $Mv_q(h) \in L^2([m,n])$ and M is a self-mapping of $L^2([m,n])$. Thus, the solution of Equation (34) is the fixed point of M. Additionally,

$$\begin{aligned} d(M_{y_l}, M_{y_s}) &= \left(\int_s^h |My_l(p) - My_s(p)|^2 dh\right)^{\frac{1}{2}} \\ &= \sum_{q=1}^{s} \left(\int_s^h |\lambda_{p,q} \int_s^h K_{p,q}(h,k)\{y_l(q) - y_s(q)\}dk|^2 dh\right)^{\frac{1}{2}} \\ &\leq \sum_{q=1}^{s} |\lambda_{p,q}| \left(\int_s^h \left\{\int_s^h (K_{p,q}(h,k))^2 dk \int_m^h |y_l(q) - y_s(q)|^2 dk\right\} dh\right)^{\frac{1}{2}} \\ &= \sum_{q=1}^{s} |\lambda_{p,q}| \left[\int_s^h \int_s^h (K_{p,q}(h,k))^2 dkdh\right]^{\frac{1}{2}} d(y_l, y_m). \end{aligned}$$

also

$$\sum_{q=1}^{s} |\lambda_{p,q}| [\int_s^h \int_s^h (K_{p,q}(h,k))^2 dkdh]^{\frac{1}{2}} \leq \mu < 1$$

Thus, M is a contractive operator. Now, from Definition 2, it follows that we have a unique solution v_q of Equation (34) satisfying Equation (2). Hence, from Theorem 4, we have $\|B_r(v_q;h) - v_q(h)\| < \epsilon$ for any $n > \frac{S}{\delta^2 \epsilon}(m+n)(2-m-n)$. This is the required result. □

4. Numerical Examples

This section consists of some numerical examples of the system of VIEs using the given technique. The absolute error of the given problems for the different values of the degree r is given in tables. The formula for the estimation of error is given by $E_{q,r} = |v_q(h) - B_r(v_q, h)|,$, where $E_{q,r}$ is the absolute error of estimating qth unknown function, $v_q(h)$ is the exact solution, and $B_r(v_q, h)$ is the approximate solution using the rth degree Bernstein polynomial.

Example 1. *Consider*

$$v_1(h) = f_1(h) + \int_0^h (h-k)^3 v_1(k)dk + \int_0^h (h-k)^2 v_2(k)dk$$

$$v_2(h) = f_2(h) + \int_0^h (h-k)^4 v_1(k)dk + \int_0^h (h-k)^3 v_2(k)dk$$

the system of VIEs. Where

$$f_1(h) = 1 + h^2 - \frac{h^3}{3} - \frac{h^4}{3}$$
$$f_2(h) = 1 + h - h^3 - \frac{h^4}{4} - \frac{h^5}{4} - \frac{h^7}{420}$$

and the exact solutions are $v_1(h) = 1 + h^2$ and $v_2(h) = 1 + h - h^3$. The approximate solution compared with the different values of the degree r is given in Table 1. The graphical representation of the approximate and exact solution is plotted in Figure 1. The comparison of the distinct nodes of the degree r clearly shows the fast convergence of the Bernstein basis function.

Table 1. Numerical results of Example 1.

h	Absolute Error for $r = 4$		Absolute Error for $r = 7$	
	$E_{1,4}(h)$	$E_{2,4}(h)$	$E_{1,7}(h)$	$E_{2,7}(h)$
0	1.01×10^{-7}	2.46×10^{-5}	3.03×10^{-14}	4.45×10^{-14}
0.1	3.96×10^{-7}	9.10×10^{-5}	2.76×10^{-14}	4.84×10^{-14}
0.2	3.29×10^{-7}	4.36×10^{-5}	2.03×10^{-14}	2.76×10^{-14}
0.3	3.75×10^{-7}	2.42×10^{-5}	5.01×10^{-15}	6.08×10^{-14}
0.4	8.91×10^{-7}	4.98×10^{-5}	2.17×10^{-14}	2.27×10^{-14}
0.5	1.91×10^{-6}	5.07×10^{-5}	1.91×10^{-14}	2.24×10^{-14}
0.6	3.15×10^{-6}	1.24×10^{-4}	8.99×10^{-15}	1.22×10^{-14}
0.7	3.99×10^{-6}	4.49×10^{-4}	4.98×10^{-14}	4.14×10^{-14}
0.8	3.52×10^{-6}	1.28×10^{-3}	6.82×10^{-14}	5.20×10^{-14}
0.9	4.72×10^{-7}	2.96×10^{-3}	2.36×10^{-14}	3.30×10^{-14}
1	6.72×10^{-6}	5.90×10^{-3}	2.61×10^{-14}	2.55×10^{-14}

Example 2. *Consider*

$$(3h-8)v_1(h) + (-2h+5)v_2(h) = f_1(h) + \int_0^h (h+k)v_1(k)dk + \int_0^h hkv_2(k)dk,$$

$$4hv_1(h) + (h-5)v_2(h) = f_2(h) + \int_0^h (2h-k)v_1(k)dk + \int_0^h (k+hk)v_2(k)dk,$$

be the system of VIEs, where

$$f_1(h) = 5e^h - 2h - 9\sin(h) - h^2e^h + 2h\cos(h) - he^h + 3h\sin(h)$$

$$f_2(h) = -3h - 4e^h + \sin(h) - h^2e^h + h\cos(h) + he^h + 4h\sin(h) - 1$$

and the exact solutions are $v_1(h) = sin(h)$ and $v_2(h) = e^h$.

The approximate solution compared with the different values of the degree r is given in Table 2. A graphical representation of the approximate and exact solution is plotted in Figure 2. The comparison between different values of the degree r clearly shows the fast convergence of the Bernstein basis function.

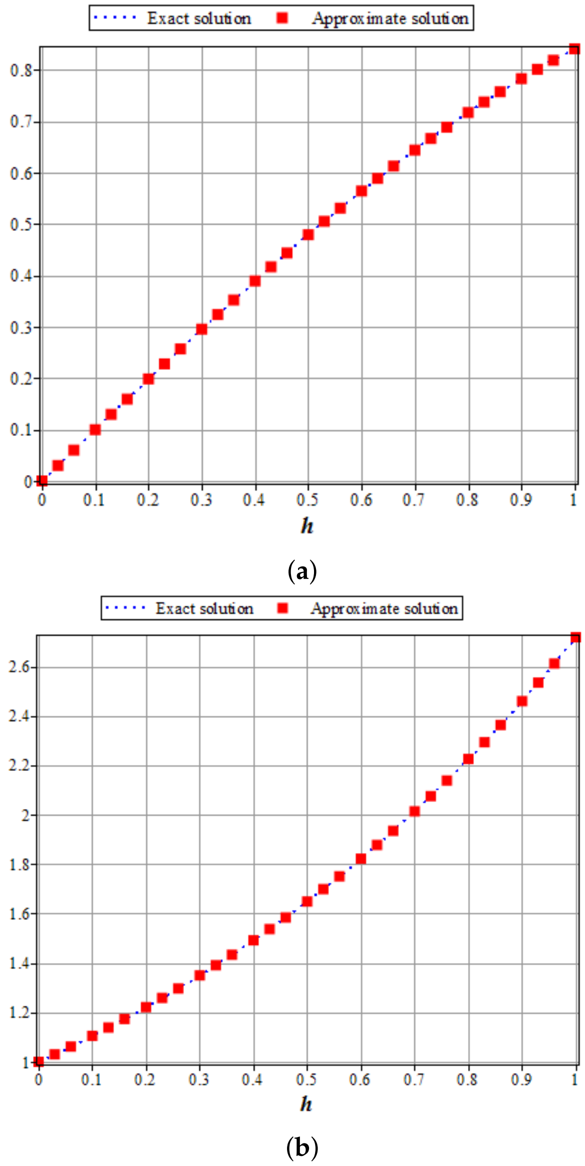


Figure 1. The approximate solution for $v_1(h)$ in (**a**) and for $v_2(h)$ in (**b**) is compared with the exact solution for the degree $r = 4$ at distict nodes of $h \in [0, 1]$.

Table 2. Numerical results of Example 2.

h	Absolute Error for $r = 4$		Absolute Error for $r = 7$	
	$E_{1,4}(h)$	$E_{2,4}(h)$	$E_{1,7}(h)$	$E_{2,7}(h)$
0	7.23×10^{-6}	1.24×10^{-5}	7.44×10^{-10}	2.77×10^{-9}
0.1	2.46×10^{-5}	4.37×10^{-5}	6.76×10^{-10}	2.51×10^{-9}
0.2	1.05×10^{-5}	1.95×10^{-5}	2.42×10^{-10}	8.61×10^{-10}
0.3	6.06×10^{-6}	1.07×10^{-5}	1.05×10^{-12}	3.35×10^{-11}
0.4	9.35×10^{-6}	1.74×10^{-5}	4.83×10^{-11}	2.27×10^{-10}
0.5	1.49×10^{-6}	2.60×10^{-6}	1.05×10^{-10}	3.15×10^{-10}
0.6	6.65×10^{-6}	1.41×10^{-5}	9.28×10^{-12}	9.41×10^{-11}
0.7	4.61×10^{-5}	1.12×10^{-5}	1.88×10^{-11}	1.70×10^{-10}
0.8	9.29×10^{-6}	1.71×10^{-5}	2.82×10^{-10}	8.84×10^{-10}
0.9	1.99×10^{-5}	4.22×10^{-5}	4.06×10^{-10}	1.33×10^{-9}
1	1.19×10^{-5}	2.06×10^{-5}	1.07×10^{-9}	2.77×10^{-9}

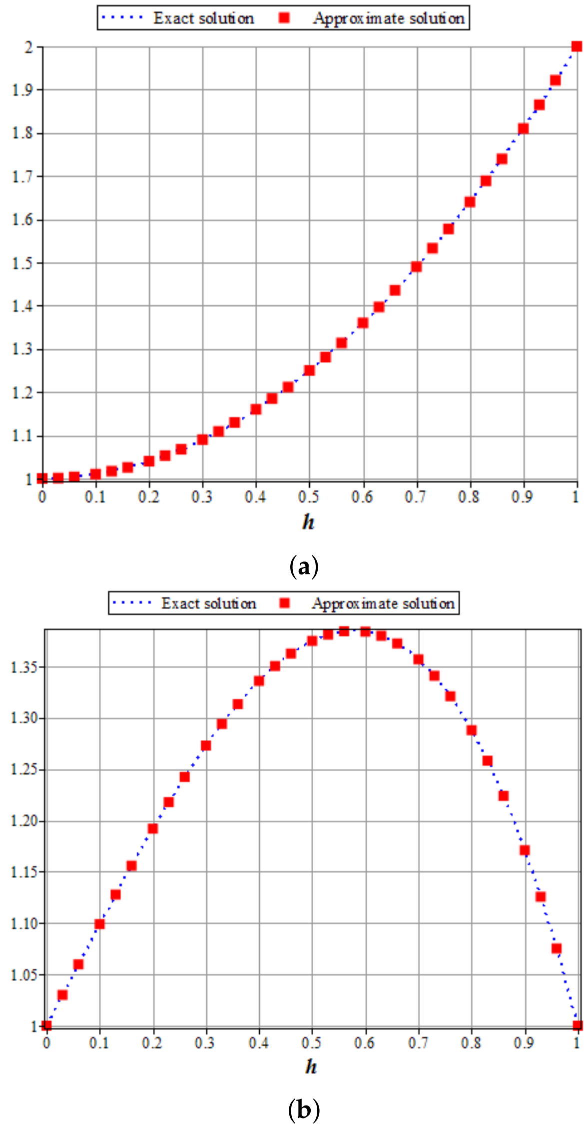


Figure 2. The approximate solution for $v_1(h)$ in (**a**) and for $v_2(h)$ in (**b**) is compared with the exact solution for the degree $r = 4$ at distinct nodes of $h \in [0, 1]$.

Example 3. *Let*

$$v_1(h) = f_1(h) - \int_0^h e^{h-k} v_1(k)dk - \int_0^h \cos(h-k) v_2(k)dk.$$

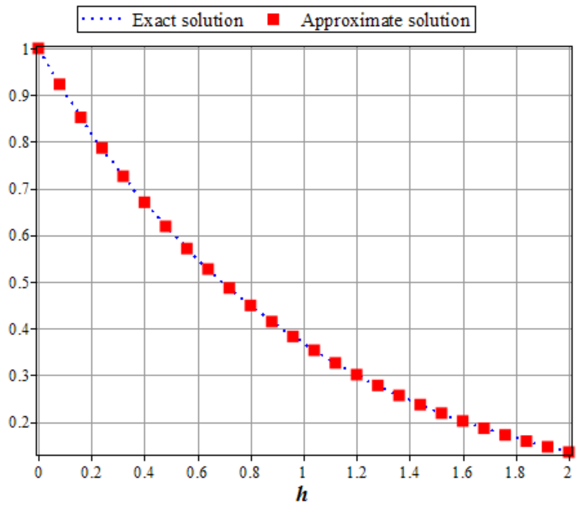
$$v_2(h) = f_2(h) - \int_0^h e^{h+k} v_1(k)dk - \int_0^h h\cos(k) v_2(k)dk.$$

be the system of VIEs. Where

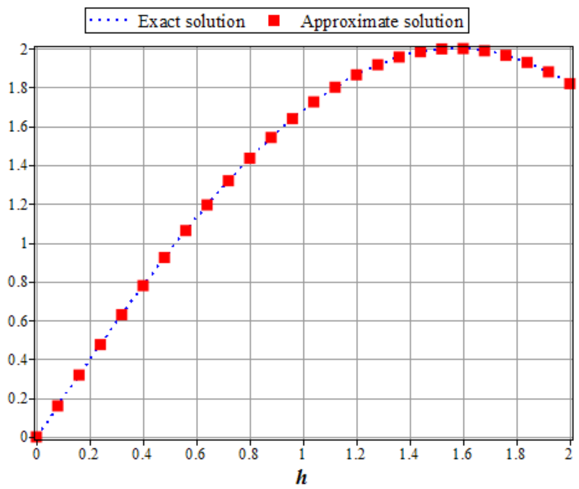
$$f_1(h) = \cosh(h) + h\sin(h)$$
$$f_2(h) = 2\sin(h) + h\sin^2(h) + e^h,$$

and the exact solutions are $v_1(h) = e^{-h}$ *,* $v_2(h) = 2\sin(h)$.

The approximate solution compared with the different values of the degree r is given in Table 3. The graphical representation of approximate and exact solution is plotted in Figure 3. The comparison between different values of the degree r clearly shows the fast convergence of the Bernstein basis function.



(**a**)



(**b**)

Figure 3. The approximate solution for $v_1(h)$ in (**a**) and for $v_2(h)$ in (**b**) is compared with the exact solution for the degree $r = 4$ at distinct nodes of $h \in [0, 2]$.

Table 3. Numerical results of Example 3.

h	Absolute Error for $r = 4$		Absolute Errors for $r = 7$	
	$E_{1,4}(h)$	$E_{2,4}(h)$	$E_{1,7}(h)$	$E_{2,7}(h)$
0	1.48×10^{-5}	1.14×10^{-4}	5.0×10^{-8}	2.6×10^{-7}
0.2	2.58×10^{-4}	1.11×10^{-3}	3.7×10^{-8}	4.2×10^{-7}
0.4	3.03×10^{-4}	6.53×10^{-4}	1.5×10^{-8}	1.9×10^{-7}
0.6	1.06×10^{-4}	3.68×10^{-4}	5.1×10^{-8}	5.4×10^{-7}
0.8	1.30×10^{-4}	7.99×10^{-4}	3.6×10^{-8}	1.0×10^{-8}
1.0	1.91×10^{-4}	3.32×10^{-5}	4.2×10^{-8}	4.6×10^{-8}
1.2	5.28×10^{-7}	1.50×10^{-3}	7.1×10^{-8}	2.0×10^{-7}
1.4	2.45×10^{-4}	1.95×10^{-3}	1.1×10^{-9}	1.97×10^{-7}
1.6	3.36×10^{-5}	2.24×10^{-3}	5.4×10^{-9}	3.8×10^{-7}
1.8	1.88×10^{-3}	1.65×10^{-2}	4.2×10^{-8}	7.8×10^{-7}
2	6.89×10^{-3}	4.83×10^{-2}	1.1×10^{-7}	2.2×10^{-6}

5. Conclusions

It is difficult to solve the system of Volterra integral equations analytically. So, finding the approximate solution of the system of Volterra integral equations on arbitrary intervals $[m, n]$ has always been a challenging problem for researchers. The proposed technique was based on approximating the unknown function with the Bernstein polynomial. The use of Bernstein polynomials is increased in different areas of mathematics due to its fast convergence. The results of the given numerical examples shows that the proposed technique is applicable and efficient. Moreover, the computed absolute error confirms its convergence and stability. It is worth pointing out that the proposed technique can find the numerical solutions of the system of Volterra integral equations on arbitrary intervals $[m, n]$.

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