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Some Essential Relations for the Quaternion Quadratic-Phase Fourier Transform

Mawardi Bahri [1] and Samsul Ariffin Abdul Karim [2,3,*]

1 Department of Mathematics, Hasanuddin University, Makassar 90245, Indonesia
2 Software Engineering Programme, Faculty of Computing and Informatics, Universiti Malaysia Sabah, Jalan UMS, Kota Kinabalu 88400, Malaysia
3 Data Technologies and Applications (DaTA) Research Group, Faculty of Computing and Informatics, Universiti Malaysia Sabah, Jalan UMS, Kota Kinabalu 88400, Malaysia
* Correspondence: samsulariffin.karim@ums.edu.my

Abstract: Motivated by the fact that the quaternion Fourier transform is a powerful tool in quaternion signal analysis, here, we study the quaternion quadratic-phase Fourier transform, which is a generalized version of the quaternion Fourier transform. We first give a definition of the quaternion quadratic-phase Fourier transform. We derive in detail some essential properties related to this generalized transformation. We explore how the quaternion quadratic-phase Fourier transform is related to the quaternion Fourier transform. It is shown that this relation allows us to obtain several versions of uncertainty principles concerning the quaternion quadratic-phase Fourier transform.

Keywords: quaternion Fourier transform; quaternion quadratic-phase Fourier transform; uncertainty principles

MSC: 11R52; 42A38; 15A66; 83A05; 35L0

1. Introduction

As is known, the quadratic-phase Fourier transform [1–6] is considered a useful tool in signal processing and has drawn great attention from some scholars in recent years. It can also be looked at as a natural generalization of several popular transformations, such as the Fourier transformation [7,8], the linear canonical transform (LCT) and the fractional Fourier transform (FrFT). On the other hand, some researchers have interest in the construction of various transformations using the quadratic-phase Fourier transform (see, e.g., [9–11]). They have also investigated the main properties of constructed transformations such as spatial shifts, multiplication, convolution and inequalities. Recently, in [12], the authors studied the Gabor quaternion quadratic-phase Fourier transform. It was shown that several properties of the proposed transformation are obtained using a direct connection between the definition of the Gabor quaternion quadratic-phase Fourier transform and the definition of the quaternion quadratic-phase Fourier transform. Therefore, it is very meaningful to study the quaternion quadratic-phase Fourier transform and investigate its properties in detail.

In the current research, we introduce a definition of the quaternion quadratic-phase Fourier transform, which can be thought as a non-trivial generalization of the quaternion Fourier transform, the quaternion linear canonical transform (QLCT) [13–17], the quaternion fractional Fourier transform (QFrFT) and the other generalized transformations. It is shown that the direct interaction between the quaternion quadratic-phase Fourier transform and the quaternion Fourier transform permits us to build some properties and novel inequalities associated with the quaternion quadratic-phase Fourier transform. We emphasize that our present work is different from the proposed method in [18], as in [18],

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the linear canonical wavelet transform was proposed, while in the present study, a new quaternion quadratic-phase Fourier transform is proposed.

This paper is arranged as follows. In Section 2, we introduce some preliminaries related to quaternion algebra which will be useful. The definition of the quaternion Fourier transform (QFT) and its uncertainty principle are included in Section 3. Section 4 is devoted to introducing the quaternion quadratic-phase Fourier transform (QQPFT) and its connection to the quaternion Fourier transform. Section 5 is devoted to the derivation of several uncertainty principles associated with the QQPFT. Lastly, Section 6 gives our conclusions.

2. Preliminary Notations

In this part, we mainly recall some basic facts on quaternion algebra and properties, which will be needed throughout this work. The quaternion algebra $\mathbb{H}$ over a real number $\mathbb{R}$ is an extension of the complex numbers in higher dimensions. An element $r \in \mathbb{H}$ is of the form [19]

$$\mathbb{H} = \{r = r_0 + \mathbf{i}r_1 + \mathbf{j}r_2 + \mathbf{k}r_3 \mid r_0, r_1, r_2, r_3 \in \mathbb{R}\}, \tag{1}$$

for which the three imaginary quaternion units $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ obey the following defining relations:

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{2}$$

For simplicity, any $r \in \mathbb{H}$ may be written as

$$r = r_0 + \boldsymbol{r} = Sc(r) + V(r). \tag{3}$$

In this case, $r_0 = Sc(r)$ and $V(r) = \mathbf{i}r_1 + \mathbf{j}r_2 + \mathbf{k}r_3$ denote the scalar and vector parts, respectively.

Due to Equation (2), the multiplication of quaternions q and r is expressed as

$$qr = q_0 r_0 - \boldsymbol{q} \cdot \boldsymbol{r} + q_0 \boldsymbol{q} + r_0 \boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{r}, \tag{4}$$

where

$$\begin{aligned} \boldsymbol{q} \cdot \boldsymbol{r} &= q_1 r_1 + q_2 r_2 + q_3 r_3, \\ \boldsymbol{q} \times \boldsymbol{r} &= \mathbf{i}\,(q_2 r_3 - q_3 r_2) + \mathbf{j}\,(q_3 r_1 - q_1 r_3) + \mathbf{k}\,(q_1 r_2 - q_2 r_1), \end{aligned}$$

and $q + r = q_0 r_0 + \mathbf{i}q_1 r_1 + \mathbf{j}q_2 r_2 + \mathbf{k}q_3 r_3$.

The conjugate of a quaternion r denoted by $\bar{r}$ is given by

$$\bar{r} = r_0 - \mathbf{i}r_1 - \mathbf{j}r_2 - \mathbf{k}r_3, \tag{5}$$

with the properties

$$\overline{qr} = \bar{r}\bar{q}, \quad \overline{r+q} = \bar{r} + \bar{q}, \quad \bar{\bar{r}} = r.$$

For every $r \in \mathbb{H}$, the following are satisfied:

$$Sc(r) = \frac{1}{2}(r + \bar{r}) \quad \text{and} \quad \mathrm{V}(r) = \frac{1}{2}(r - \bar{r}). \tag{6}$$

The module (norm) of the quaternion q can be defined as

$$|r| = \sqrt{r\bar{r}} = \sqrt{r_0^2 + r_1^2 + r_2^2 + r_3^2}. \tag{7}$$

One can easily verify that for every $q, r, p \in \mathbb{H}$, the following holds:

$$Sc(q) \le |q|, \quad |\boldsymbol{q}| = |V(q)| \le |q|, \quad Sc(qpr) = Sc(prq) = Sc(qpr). \tag{8}$$

It is straightforward to see that

$$|qr| = |q||r|, \tag{9}$$

and

$$|q + r| \leq |q| + |r|. \tag{10}$$

The inverse of a non-zero quaternion $r \in \mathbb{H}$ is expressed as

$$r^{-1} = \frac{\bar{r}}{|r|^2}. \tag{11}$$

This will lead to the following:

Lemma 1. *Let q be a nonzero quaternion. For every $r \in \mathbb{H}$, one has*

$$\left|\frac{r}{q}\right| = \frac{|r|}{|q|}. \tag{12}$$

Proof. Since $|\bar{q}| = |q|$, under Equations (9) and (11), we have

$$\left|\frac{r}{q}\right| = \left|r\frac{\bar{q}}{|q|^2}\right| = \frac{1}{|q|^2}|r\bar{q}| = \frac{|r|}{|q|},$$

which gives the required result. □

Furthermore, one may consider the inner product for two quaternion functions f and g as follows:

$$(f, g) = \int_{\mathbb{R}^2} f(\boldsymbol{t})\overline{g(\boldsymbol{t})}\, d\boldsymbol{t}, \quad d\boldsymbol{t} = dt_1 dt_2. \tag{13}$$

with the scalar product

$$\langle f, g\rangle = Sc(f, g). \tag{14}$$

Set

$$\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} = \left(\int_{\mathbb{R}^2} |f(\boldsymbol{t})|^2\, d\boldsymbol{t}\right)^{1/2}. \tag{15}$$

3. Quaternion Fourier Transform

This part begins by defining the quaternion Fourier transform (QFT). Since the quaternion product is not always commutative in general, there are three different kinds of QFTs: the so-called two-sided QFT, right-sided QFT and left-sided QFT. We focus on the introduction of the two-sided quaternion Fourier transform (or the QFT for short). We collect its important properties, including the inversion formula and uncertainty principles, which will be needed later. More details of the QFT properties, including its uncertainty principle, are referred to in [20–25]:

Definition 1. *Let $f \in L^1(\mathbb{R}^2;\mathbb{H}) \cap L^2(\mathbb{R}^2;\mathbb{H})$. The definition of the two-sided quaternion Fourier transform of f is described through*

$$\mathcal{F}_H\{f\}(\boldsymbol{\xi}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-\mathbf{i}\xi_1 t_1} f(\boldsymbol{t}) e^{-\mathbf{j}\xi_2 t_2}\, d\boldsymbol{t}, \tag{16}$$

where $\boldsymbol{\xi}, \boldsymbol{t} \in \mathbb{R}^2$.

The reconstruction formula of the QFT defined above is computed by using the following definition:

Definition 2. *Let $f \in L^1(\mathbb{R}^2;\mathbb{H})$ and $\mathcal{F}_H\{f\} \in L^1(\mathbb{R}^2;\mathbb{H})$. The inverse quaternion Fourier transform for f over $\mathbb{R}^2$ is evaluated as follows:*

$$f(\boldsymbol{t}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mathbf{i}\xi_1 t_1}\mathcal{F}_H\{f\}(\boldsymbol{\xi})\, e^{\mathbf{j}\xi_2 t_2}\, d\boldsymbol{\xi}. \tag{17}$$

We can easily verify the following result:

Lemma 2. *If $f(\boldsymbol{t}), \boldsymbol{t}f(\boldsymbol{t}) \in L^2(\mathbb{R}^2;\mathbb{H})$, then $\frac{\partial^2}{\partial\omega_1\partial\xi_2}\mathcal{F}_H\{f\}(\boldsymbol{\xi})$ exists and is given by*

$$\frac{\partial^2}{\partial\xi_1\partial\xi_2}\mathcal{F}_H\{f\}(\boldsymbol{\xi}) = \mathbf{i}\mathcal{F}\{f_\tau(\boldsymbol{t})\}(\boldsymbol{\xi})\mathbf{j}, \tag{18}$$

where $f_\tau(\boldsymbol{t}) = t_1 f(\boldsymbol{t}) t_2$.

Proof. Due to the relation in Equation (16), we have

$$\begin{aligned}\frac{\partial}{\partial\xi_1}\mathcal{F}_H\{f\}(\boldsymbol{\xi}) &= \frac{\partial}{\partial\xi_1}\int_{\mathbb{R}^2} e^{-\mathbf{i}\xi_1 t_1} f(\boldsymbol{t}) e^{-\mathbf{j}\xi_2 t_2}\, d\boldsymbol{t} \\ &= \int_{\mathbb{R}^2} -it_1 e^{-\mathbf{i}\xi_1 t_1} f(\boldsymbol{t}) e^{-j\xi_2 t_2}\, d\boldsymbol{t} \\ &= -\mathbf{i}\int_{\mathbb{R}^2} e^{-i\xi_1 t_1} t_1 f(\boldsymbol{t}) e^{-\mathbf{j}\xi_2 t_2}\, d\boldsymbol{t}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\frac{\partial^2}{\partial\xi_1\partial\xi_2}\mathcal{F}_H\{f\}(\boldsymbol{\xi}) &= \frac{\partial}{\partial\xi_2}\left[\frac{\partial}{\partial\xi_1}\mathcal{F}_H\{f\}(\boldsymbol{\xi})\right] \\ &= \frac{\partial}{\partial\xi_2}\left[-\mathbf{i}\int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 t_1} t_1 f(\boldsymbol{t}) e^{-\mathbf{j}\xi_2 t_2}\, d\boldsymbol{t}\right] \\ &= -\mathbf{i}\int_{\mathbb{R}^2} e^{-\mathbf{i}\xi_1 t_1} t_1 f(\boldsymbol{t})(-\mathbf{j}) t_2 e^{-\mathbf{j}\xi_2 t_2}\, d\boldsymbol{t} \\ &= -\mathbf{i}\int_{\mathbb{R}^2} e^{-\mathbf{i}\xi_1 t_1} t_1 f(\mathbf{t}) t_2 e^{-\mathbf{j}\xi_2 t_2} d\boldsymbol{t}(-\mathbf{j}) \\ &= -\mathbf{i}\mathcal{F}\{f_\tau(\boldsymbol{t})\}(\boldsymbol{\xi})(-\mathbf{j}).\end{aligned}$$

The proof is complete. □

Useful results for the QFT defined by Equation (16) include the inequalities demonstrated by the following formulas:

Theorem 1 (Heisenberg's inequality for QFT [26])**.** *For all $f \in L^2(\mathbb{R}^2;\mathbb{H})$ with $\mathcal{F}_H\{f\} \in L^2(\mathbb{R}^2;\mathbb{H})$, the following inequality holds:*

$$\int_{\mathbb{R}^2} t_k^2 |f(\boldsymbol{t})|^2\, d\boldsymbol{t} \int_{\mathbb{R}^2} \xi_k^2 |\mathcal{F}_H\{f\}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} \geq \frac{1}{4}\left(\int_{\mathbb{R}^2} |f(\boldsymbol{t})|^2\, d\boldsymbol{t}\right)^2, \quad k = 1,2. \tag{19}$$

Theorem 2 (QFT Sharp Hausdorff–Young Inequality [20])**.** *Let $1 \leq r \leq 2$ and s be such that $\frac{1}{r} + \frac{1}{s} = 1$. Then, for any $f \in L^r(\mathbb{R}^2;\mathbb{H})$, the following inequality holds:*

$$\|\mathcal{F}_H\{f\}\|_{L^s(\mathbb{R}^2;\mathbb{H})} \leq C_r^2 \|f\|_{L^r(\mathbb{R}^2;\mathbb{H})}, \tag{20}$$

where

$$C_r = \left(r^{1/r} s^{-1/s}\right)^{1/2}. \tag{21}$$

Theorem 3 (QFT Pitt's inequality [24])**.** *Let $f \in \mathcal{S}(\mathbb{R}^2;\mathbb{H})$ and $0 \le \alpha < 2$. Then, we have*

$$\int_{\mathbb{R}^2} |\boldsymbol{\xi}|^{-\alpha} |\mathcal{F}_H\{f\}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \le M(\alpha) \int_{\mathbb{R}^2} |\boldsymbol{t}|^{\alpha} |f(\boldsymbol{t})|^2 \, d\boldsymbol{t}, \tag{22}$$

where

$$M(\alpha) = \pi^{\alpha} \left[\frac{\Gamma\left(\frac{2-\alpha}{4}\right)}{\Gamma\left(\frac{2+\alpha}{4}\right)} \right]^2. \tag{23}$$

Here, $\mathcal{S}(\mathbb{R}^2;\mathbb{H})$ is the quaternion Schwartz space and $\Gamma(\cdot)$ is the well-known gamma function.

4. Quaternion Quadratic-Phase Fourier Transform (QQPFT)

The main purpose of this section is to derive the useful properties of the quaternion quadratic-phase Fourier transform (QQPFT). We also study its fundamental relationship to the quaternion Fourier transform and adopt it to provide the proof of the uncertainty principles related to the quaternion quadratic-phase Fourier transform.

4.1. QQPFT Definitions

Let us start by introducing the three types of definitions for the quaternion quadratic-phase Fourier transform:

Definition 3. *Let $Q_1 = (A_1, B_1, C_1, D_1)$ and $Q_2 = (A_2, B_2, C_2, D_2)$ be the given parameter sets. The two-sided quaternion quadratic-phase Fourier transform of a signal $f \in L^1(\mathbb{R}^2;\mathbb{H}) \cap L^2(\mathbb{R}^2;\mathbb{H})$ such that $B_1 \neq 0$ and $B_2 \neq 0$ is defined through*

$$\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi}) = \frac{1}{\sqrt{(2\pi)^2}} \int_{\mathbb{R}^2} K_{Q_1}(t_1, \xi_1) f(\boldsymbol{t}) K_{Q_2}(t_2, \xi_2) \, d\boldsymbol{t}. \tag{24}$$

Definition 4. *Let $Q_1 = (A_1, B_1, C_1, D_1)$ and $Q_2 = (A_2, B_2, C_2, D_2)$ be the given parameter sets. The right-sided quaternion quadratic-phase Fourier transform of a signal $f \in L^1(\mathbb{R}^2;\mathbb{H}) \cap L^2(\mathbb{R}^2;\mathbb{H})$ such that $B_1 \neq 0$ and $B_2 \neq 0$ is defined through*

$$\mathcal{Q}^R_{Q_1,Q_2}\{f\}(\boldsymbol{\xi}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(\boldsymbol{t}) K_{Q_1}(t_1, \xi_1) K_{Q_2}(t_2, \xi_2) \, d\boldsymbol{t}. \tag{25}$$

Definition 5. *Let $Q_1 = (A_1, B_1, C_1, D_1)$ and $Q_2 = (A_2, B_2, C_2, D_2)$ be the given parameter sets. The left-sided quaternion quadratic-phase Fourier transform of a signal $f \in L^1(\mathbb{R}^2;\mathbb{H}) \cap L^2(\mathbb{R}^2;\mathbb{H})$ such that $B_1 \neq 0$ and $B_2 \neq 0$ is defined through*

$$\mathcal{Q}^L_{Q_1,Q_2}\{f\}(\boldsymbol{\xi}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} K_{Q_1}(t_1, \xi_1) K_{Q_2}(t_2, \xi_2) f(\boldsymbol{t}) \, d\boldsymbol{t}. \tag{26}$$

Here, we have

$$K_{Q_1}(t_1, \xi_1) = \frac{1}{\sqrt{2\pi}} e^{-\mathbf{i}\left(A_1 t_1^2 + B_1 t_1 \xi_1 + C_1 \xi_1^2 + D_1 t_1 + E_1 \xi_1\right)}, \tag{27}$$

and

$$K_{Q_2}(t_2, \xi_2) = \frac{1}{\sqrt{2\pi}} e^{-\mathbf{j}\left(A_2 t_2^2 + B_2 t_2 \xi_2 + C_2 \xi_2^2 + D_2 t_2 + E_2 \xi_2\right)}. \tag{28}$$

Our main work is solely focused on the two-sided quaternion quadratic-phase Fourier transform (QQPFT).

Some notable special cases include the following:

- When $Q_1 = (0,1,0,0,0)$ and $Q_2 = (0,1,0,0,0)$, the above definition boils down to the QFT definition in Equation (16).
- For the parameter sets $Q_1 = (\cot\frac{\theta}{2}, -\csc\theta, \cot\frac{\theta}{2}, 0, 0)$ and $Q_2 = (\cot\frac{\vartheta}{2}, -\csc\vartheta, \cot\frac{\vartheta}{2}, 0, 0)$, multiplying the left side of Equation (24) by $\sqrt{1-\mathbf{i}\cot\theta}$ and the right side of Equation (24) by $\sqrt{1-\mathbf{j}\cot\vartheta}$ reduces to the quaternion fractional Fourier transform.

Some essential properties of the QQPFT are shown in the following theorem:

Theorem 4. *If the quaternion function f belongs to $L^1(\mathbb{R}^2;\mathbb{H})$, then $\mathcal{Q}_{Q_1,Q_2}\{f\}$ is continuous on $\mathbb{R}^2$.*

Proof. Applying the QQPFT definition (Equation (24)) shows that for every $\boldsymbol{h} \in \mathbb{R}^2$, we have

$$\begin{aligned}
&\left|\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi}+\boldsymbol{h}) - \mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\right| \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-\mathbf{i}\left(A_1t_1^2+B_1t_1(\xi_1+h_1)+C_1(\xi_1+h_1)^2+D_1t_1+E_1(\xi_1+h_1)\right)} f(\boldsymbol{t}) \\
&\quad\times e^{-\mathbf{j}\left(A_2t_2^2+B_2t_2(\xi_2+h_2)+C_2(\xi_2+h_2)^2+D_2t_2+E_2(\xi_2+h_2)\right)}\,d\boldsymbol{t} \\
&\quad - \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-\mathbf{i}\left(A_1t_1^2+B_1t_1\xi_1+C_1\xi_1^2+D_1t_1+E_1\xi_1\right)} f(\boldsymbol{t})\, e^{-\mathbf{j}\left(A_2t_2^2+B_2t_2\xi_2+C_2\xi_2^2+D_2t_2+E_2\xi_2\right)}\,d\boldsymbol{t} \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-\mathbf{i}\left(A_1t_1^2+B_1t_1\xi_1+C_1\xi_1^2+D_1t_1+E_1\xi_1\right)} \left(e^{-\mathbf{i}\left(B_1t_1h_1+2C_1\xi_1h_1+C_1h_1^2+E_1h_1\right)} f(\boldsymbol{t})\right. \\
&\quad\left.\times e^{-\mathbf{j}\left(B_2t_2h_2+2C_2\xi_2h_2+C_2h_2^2+E_2h_2\right)} - f(\boldsymbol{t})\right) e^{-\mathbf{j}\left(A_2t_2^2+B_2t_2\xi_2+C_2\xi_2^2+D_2t_2+E_2\xi_2\right)}\,d\boldsymbol{t} \\
&\leq \frac{1}{2\pi}\int_{\mathbb{R}^2} \left| e^{-\mathbf{i}\left(A_1t_1^2+B_1t_1\xi_1+C_1\xi_1^2+D_1t_1+E_1\xi_1\right)} \left(e^{-\mathbf{i}\left(B_1t_1h_1+2C_1\xi_1h_1+C_1h_1^2+E_1h_1\right)} f(\boldsymbol{t})\right.\right. \\
&\quad\left.\left.\times e^{-\mathbf{j}\left(B_2t_2h_2+2C_2\xi_2h_2+C_2h_2^2+E_2h_2\right)} - f(\boldsymbol{t})\right) e^{-\mathbf{j}\left(A_2t_2^2+B_2t_2\xi_2+C_2\xi_2^2+D_2t_2+E_2\xi_2\right)} \right| d\boldsymbol{t}. \qquad (29)
\end{aligned}$$

Applying the triangle inequality for the quaternion described in Equation (10) results in

$$\left|\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi}+\boldsymbol{h}) - \mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\right| \leq \frac{1}{2\pi}\int_{\mathbb{R}^2} |f(\boldsymbol{t})|\,d\boldsymbol{t}. \qquad (30)$$

From Equation (29), we find that

$$\lim_{\boldsymbol{h}\to\mathbf{0}} \left|\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi}+\boldsymbol{h}) - \mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\right| = \mathbf{0}. \qquad (31)$$

Using the relations in Equations (30) and (31) with the Lebesgue-dominated convergence theorem, we deduce that $\mathcal{Q}_{Q_1,Q_2}\{f\}$ is continuous on $\mathbb{R}^2$. □

A generalization of the Riemann–Lebesgue lemma in the setting of the QQPFT is demonstrated by the following result:

Theorem 5. *Let the quaternion function belong to $L^1(\mathbb{R}^2;\mathbb{H})$. Then, we have*

$$\lim_{|\xi_1|\to\infty} \left|\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\right| = \mathbf{0}, \qquad (32)$$

uniformly in ξ_2, and

$$\lim_{|\xi_2|\to\infty} \left|\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\right| = \mathbf{0}, \tag{33}$$

uniformly in ξ_1.

Proof. We derive only the first part, and the proof of the remaining part is quite similar. It is straightforward to verify that

$$e^{-\mathbf{i}B_1\xi_1 t_1} = -e^{-\mathbf{i}B_1\xi_1(t_1+\frac{\pi}{B_1\xi_1})}. \tag{34}$$

Applying Equaiton (34) results in

$$\begin{aligned}&\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\\ &= -\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-\mathbf{i}\left(A_1t_1^2+B_1\xi_1(t_1+\frac{\pi}{B_1\xi_1})+C_1\xi_1^2+D_1t_1+E_1\xi_1\right)} f(\boldsymbol{t})e^{-\mathbf{j}\left(A_2t_2^2+B_2t_2\xi_2+C_2\xi_2^2+D_2t_2+E_2\xi_2\right)}\,d\boldsymbol{t}.\end{aligned} \tag{35}$$

The relation in Equation (35) can be rewritten in the form

$$\begin{aligned}&\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\\
&= -\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-\mathbf{i}\left(A_1(t_1-\frac{\pi}{B_1\xi_1})^2+B_1\xi_1t_1+C_1\xi_1^2+D_1(t_1-\frac{\pi}{B_1\xi_1})+E_1\xi_1\right)} f(t_1-\frac{\pi}{B_1\xi_1},t_2)\\
&\quad\times e^{-\mathbf{j}\left(A_2t_2^2+B_2t_2\xi_2+C_2\xi_2^2+D_2t_2+E_2\xi_2\right)}\,d\boldsymbol{t}\\
&= \frac{1}{4\pi}\Bigg[\int_{\mathbb{R}^2} e^{-\mathbf{i}\left(A_1t_1^2+B_1t_1\xi_1+C_1\xi_1^2+D_1t_1+E_1\xi_1\right)} f(\boldsymbol{t})e^{-\mathbf{j}\left(A_2t_2^2+B_2t_2\xi_2+C_2\xi_2^2+D_2t_2+E_2\xi_2\right)}\,d\boldsymbol{t}\,d\boldsymbol{t}\\
&\quad-\int_{\mathbb{R}^2} e^{-\mathbf{i}\left(A_1(t_1-\frac{\pi}{B_1\xi_1})^2+B_1\xi_1t_1+C_1\xi_1^2+D_1(t_1-\frac{\pi}{B_1\xi_1})+E_1\xi_1\right)} f(t_1-\frac{\pi}{B_1\xi_1},t_2)\\
&\quad\times e^{-\mathbf{j}\left(A_2t_2^2+B_2t_2\xi_2+C_2\xi_2^2+D_2t_2+E_2\xi_2\right)}\,d\boldsymbol{t}\Bigg]\\
&= \frac{1}{4\pi}\int_{\mathbb{R}^2} e^{-\mathbf{i}\left(A_1t_1^2+B_1t_1\xi_1+C_1\xi_1^2+D_1t_1+E_1\xi_1\right)}\\
&\quad\times\left(f(\boldsymbol{t}) - e^{\mathbf{i}A_1\left(-\frac{2\pi t_1}{B_1\xi_1}+(\frac{\pi}{B_1\xi_1})^2+\mathbf{i}D_1\left(-\frac{2\pi t_1}{B_1\xi_1}+(\frac{\pi}{B_1\xi_1})^2\right)\right.} f(t_1-\frac{\pi}{B_1\xi_1},t_2)\right)\\
&\quad\times e^{-\mathbf{j}\left(A_2t_2^2+B_2t_2\xi_2+C_2\xi_2^2+D_2t_2+E_2\xi_2\right)}\,d\boldsymbol{t}.\end{aligned}$$

This yields

$$\begin{aligned}&\lim_{|\xi_1|\to\infty}\left|\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\right|\\
&\qquad\le \frac{1}{4\pi}\lim_{|\xi_1|\to\infty}\int_{\mathbb{R}^2}\left|f(\boldsymbol{t}) - e^{\mathbf{i}(A_1+B_1)\left(-\frac{2\pi t_1}{B_1\xi_1}+(\frac{\pi}{B_1\xi_1})^2\right)} f(t_1-\frac{\pi}{B_1\xi_1},t_2)\right|d\boldsymbol{t} = \mathbf{0},\end{aligned}$$

uniformly in ξ_2. This ends the proof of the theorem. □

4.2. Relation between the QQPFT and QFT

In order to obtain the other properties of the quaternion quadratic-phase Fourier transform, we need to introduce the direct interaction between the quaternion quadratic-phase Fourier transform and the quaternion Fourier transform as expressed below:

$$
\begin{aligned}
&\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi}) \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-\mathbf{i}\left(A_1t_1^2+B_1t_1\xi_1+C_1\xi_1^2+D_1t_1+E_1\xi_1\right)} f(\boldsymbol{t})\, e^{-\mathbf{j}\left(A_2t_2^2+B_2t_2\xi_2+C_2\xi_2^2+D_2t_2+E_2\xi_2\right)}\, d\boldsymbol{t} \\
&= e^{-\mathbf{i}(C_1\xi_1^2+E_1\xi_1)}\left\{\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-\mathbf{i}B_1t_1\xi_1}\left(e^{-\mathbf{i}A_1t_1^2}e^{-\mathbf{i}D_1t_1}f(\boldsymbol{t})e^{-\mathbf{j}D_2t_2}e^{-\mathbf{j}A_2t_2^2}\right)\right. \\
&\quad \left.\times\, e^{-\mathbf{j}B_2t_2\xi_2}\, d\boldsymbol{t}\right\} e^{-\mathbf{j}(C_2\xi_2^2+E_2\xi_2)} \\
&= e^{-\mathbf{i}(C_1\xi_1^2+E_1\xi_1)}\left\{\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-\mathbf{i}B_1t_1\xi_1}\check{f}(\boldsymbol{t})e^{-\mathbf{j}B_2t_2\xi_2}\, d\boldsymbol{t}\right\} e^{-\mathbf{j}(C_2\xi_2^2+E_2\xi_2)} \\
&= e^{-\mathbf{i}(C_1\xi_1^2+E_1\xi_1)}\mathcal{F}_{\mathbb{H}}\{\check{f}\}(B_1\xi_1,B_2\xi_2)e^{-\mathbf{j}(C_2\xi_2^2+E_2\xi_2)},
\end{aligned} \tag{36}
$$

where

$$\check{f}(\boldsymbol{t}) = e^{-\mathbf{i}D_1t_1}e^{-\mathbf{i}A_1t_1^2}f(\boldsymbol{t})e^{-\mathbf{j}A_2t_2^2}e^{-\mathbf{j}D_2t_2}. \tag{37}$$

and

$$|\check{f}(\boldsymbol{t})| = |f(\boldsymbol{t})|. \tag{38}$$

From the relation in Equation (36), we immediately obtain

$$e^{\mathbf{i}(C_1\xi_1^2+E_1\xi_1)}\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})e^{\mathbf{j}(C_2\xi_2^2+E_2\xi_2)} = \mathcal{F}_{\mathbb{H}}\{\check{f}\}(B_1\xi_1,B_2\xi_2). \tag{39}$$

Moreover, we have

$$\left|\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\right| = \left|\mathcal{F}_{\mathbb{H}}\{\check{f}\}(B_1\xi_1,B_2\xi_2)\right|. \tag{40}$$

Now, we formulate the inversion theorem associated with the QQPFT as shown in the next result:

Theorem 6. *If $f \in L^1(\mathbb{R}^2;\mathbb{H})$ and $\mathcal{Q}_{Q_1,Q_2}\{f\} \in L^1(\mathbb{R}^2;\mathbb{H})$, then the reconstruction formula of the QQPFT is given by*

$$f(\boldsymbol{t}) = |B_1B_2|\int_{\mathbb{R}^2}\overline{K_{Q_1}(t_1,\xi_1)}\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\overline{K_{Q_2}(t_2,\xi_2)}\, d\boldsymbol{\xi}. \tag{41}$$

Proof. In light of the reconstruction formula for the QFT defined by Equation (17), we find that

$$
\begin{aligned}
\check{f}(\boldsymbol{t}) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mathbf{i}\xi_1t_1}\mathcal{F}_H\{\check{f}\}(\boldsymbol{\xi})\, e^{\mathbf{j}\xi_2t_2}\, d\boldsymbol{\xi} \\
&= |B_1B_2|\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mathbf{i}B_1\xi_1t_1}\mathcal{F}_H\{\check{f}\}(B_1\xi_1,B_2\xi_2)\, e^{\mathbf{j}B_2\xi_2t_2}\, d\boldsymbol{\xi} \\
&= |B_1B_2|\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mathbf{i}B_1\xi_1t_1}e^{\mathbf{i}(C_1\xi_1^2+E_1\xi_1)}\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})e^{\mathbf{j}(C_2\xi_2^2+E_2\xi_2)}\, e^{\mathbf{j}B_2\xi_2t_2}\, d\boldsymbol{\xi}.
\end{aligned} \tag{42}
$$

From Equation (37), we further obtain

$$
\begin{aligned}
&e^{-\mathbf{i}D_1t_1}e^{-\mathbf{i}A_1t_1^2}f(\boldsymbol{t})e^{-\mathbf{j}A_2t_2^2}e^{-\mathbf{j}D_2t_2} \\
&= |B_1B_2|\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mathbf{i}B_1\xi_1t_1}e^{\mathbf{i}(C_1\xi_1^2+E_1\xi_1)}\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})e^{\mathbf{j}(C_2\xi_2^2+E_2\xi_2)}\, e^{\mathbf{j}B_2\xi_2t_2}\, d\boldsymbol{\xi}.
\end{aligned} \tag{43}
$$

This will lead to

$$
\begin{aligned}
f(\boldsymbol{t}) &= |B_1B_2|\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mathbf{i}D_1t_1}e^{\mathbf{i}A_1t_1^2}e^{\mathbf{i}B_1\xi_1t_1}e^{\mathbf{i}(C_1\xi_1^2+E_1\xi_1)}\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi}) \\
&\quad\times e^{\mathbf{j}(C_2\xi_2^2+E_2\xi_2)}\, e^{\mathbf{j}B_2\xi_2t_2}e^{\mathbf{j}A_2t_2^2}e^{\mathbf{j}D_2t_2}\, d\boldsymbol{\xi} \\
&= |B_1B_2|\int_{\mathbb{R}^2}\overline{K_{Q_1}(t_1,\xi_1)}\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\overline{K_{Q_2}(t_2,\xi_2)}\, d\boldsymbol{\xi},
\end{aligned}
$$

The proof is complete. □

In the following theorem, we formulate the scalar part of Parseval's formula for the QQPFT, which will be useful for deriving the uncertainty principles concerning the proposed transformation:

Theorem 7 (QQPFT Parseval's formula)**.** *Let $f,g\in L^2(\mathbb{R}^2;\mathbb{H})$. Then, it holds that*

$$\langle f,g\rangle = |B_1B_2|\langle \mathcal{Q}_{Q_1,Q_2}\{f\},\mathcal{Q}_{Q_1,Q_2}\{g\}\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{44}$$

In particular, we have

$$\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} = |B_1B_2|\|\mathcal{Q}_{Q_1,Q_2}\{f\}\|_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{45}$$

Proof. According to Parseval's formula for the QFT defined by Equation (16), for the functions $\check{f}$ and $\check{g}$, we have

$$
\begin{aligned}
\langle \check{f},\check{g}\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} &= \langle \mathcal{F}_{\mathbb{H}}\{\check{f}\},\mathcal{F}_{\mathbb{H}}\{\check{g}\}\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} \\
&= Sc\left(\int_{\mathbb{R}^2}\mathcal{F}_{\mathbb{H}}\{\check{f}\}(\boldsymbol{\xi})\overline{\mathcal{F}_{\mathbb{H}}\{\check{g}\}(\boldsymbol{\xi})}\, d\boldsymbol{\xi}\right) \\
&= |B_1B_2|Sc\left(\int_{\mathbb{R}^2}\mathcal{F}_{\mathbb{H}}\{\check{f}\}(B_1\boldsymbol{\xi})\overline{\mathcal{F}_{\mathbb{H}}\{\check{g}\}(B_2\boldsymbol{\xi})}\, d\boldsymbol{\xi}\right).
\end{aligned} \tag{46}
$$

Using the relation in Equation (39), we infer that

$$
\begin{aligned}
&\langle \check{f},\check{g}\rangle \\
&= |B_1B_2|\int_{\mathbb{R}^2} Sc\Big(e^{\mathbf{i}(C_1\xi_1^2+E_1\xi_1)}\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})e^{\mathbf{j}(C_2\xi_2^2+E_2\xi_2)}e^{-\mathbf{j}(C_2\xi_2^2+E_2\xi_2)} \\
&\times \overline{\mathcal{Q}_{Q_1,Q_2}\{g\}(\boldsymbol{\xi})}e^{-\mathbf{i}(C_1\xi_1^2+E_1\xi_1)}\Big)\, d\boldsymbol{\xi} \\
&= |B_1B_2|\int_{\mathbb{R}^2} Sc\Big(\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\overline{\mathcal{Q}_{Q_1,Q_2}\{g\}(\boldsymbol{\xi})}\Big)\, d\boldsymbol{\xi}.
\end{aligned} \tag{47}
$$

The proof is complete. □

Remark 1. *It should be observed that for the right-sided quaternion quadratic-phase Fourier transform, the full version of Parseval's formula is valid:*

$$(f,g) = |B_1B_2|\big(\mathcal{Q}_{Q_1,Q_2}\{f\},\mathcal{Q}_{Q_1,Q_2}\{g\}\big)_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{48}$$

5. Inequalities for QQPFT

It is known that Heisenberg's uncertainty principle in quantum physics tells us that the position and velocity (or momentum) of a particle cannot be determined at exactly the same time. In harmonic analysis especially, the uncertainty principle explains a relation of the function to its Fourier transform. More precisely, it stated that a nonzero function and its Fourier transformation cannot simultaneously concentrate around points. Inspired by these facts, we extend several versions of the uncertainty principles in the context of the quaternion quadratic-phase Fouurier transform (QQPFT). First, we obtain a sharp Hausdorff–Young inequality for the QQPFT as shown in the following theorem:

Theorem 8 (QQPFT Sharp Hausdorff-Young)**.** *Under the same conditions as in Theorem 2, we have*

$$|\mathcal{Q}_{Q_1,Q_2}\{f\}\|_{L^s(\mathbb{R}^2;\mathbb{H})} \leq \frac{C_r^2}{|B_1B_2|^{\frac{1}{s}}}\|f\|_{L^r(\mathbb{R}^2;\mathbb{H})}. \tag{49}$$

where C_r is defined by Equation (21).

Proof. It directly follows from the relation in Equation (40) that

$$\begin{aligned} \|\mathcal{Q}_{Q_1,Q_2}\{f\}\|_{L^s(\mathbb{R}^2;\mathbb{H})} &= \left(\int_{\mathbb{R}^2}|\mathcal{Q}_{Q_1,Q_2}\{f\}(\xi_1,\xi_2)|\,d\boldsymbol{\xi}\right)^{\frac{1}{s}} \\ &= \left(\int_{\mathbb{R}^2}\left|\mathcal{F}_H\{\check{f}\}(B_1\xi_1,B_2\xi_2)\right|^s d\boldsymbol{\xi}\right)^{\frac{1}{s}} \\ &= \left(\frac{1}{|B_1B_2|}\int_{\mathbb{R}^2}\left|\mathcal{F}_H\{\check{f}\}(\xi_1,\xi_2)\right|^s d\boldsymbol{\xi}\right)^{\frac{1}{s}} \\ &= \frac{1}{|B_1B_2|^{\frac{1}{s}}}\|\mathcal{F}_H\check{f}\|_{L^s(\mathbb{R}^2;\mathbb{H})}. \end{aligned} \tag{50}$$

Applying the relations in Equations (20) and (38) results in

$$\begin{aligned} \|\mathcal{Q}_{Q_1,Q_2}\{f\}\|_{L^s(\mathbb{R}^2;\mathbb{H})} &\leq \frac{C_r^2}{|B_1B_2|^{\frac{1}{s}}}\|\check{f}\|_{L^r(\mathbb{R}^2;\mathbb{H})} \\ &= \frac{C_r^2}{|B_1B_2|^{\frac{1}{s}}}\|f\|_{L^r(\mathbb{R}^2;\mathbb{H})}, \end{aligned} \tag{51}$$

The proof is complete. □

We further formulate Pitt's inequality for the QQPFT, which is an extension of Pitt's inequality for the QFT in Theorem 3. It seems that this result is an improved version of Pitt's inequality for the QFT:

Theorem 9 (QQPFT Pitt's inequality)**.** *Under the same assumptions as in Theorem 3, the following holds:*

$$\int_{\mathbb{R}^2}|\boldsymbol{\xi}|^{-\alpha}\left|\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} \leq M(\alpha)|B_1B_2|^{\alpha-1}\int_{\mathbb{R}^2}|\boldsymbol{t}|^{\alpha}\left|f(\boldsymbol{t})\right|^2 d\boldsymbol{t}. \tag{52}$$

Proof. Including Equation (37) into both sides of Equaiton (22) yields

$$\int_{\mathbb{R}^2}|\boldsymbol{\xi}|^{-\alpha}|\mathcal{F}_H\{\check{f}\}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \leq M(\alpha)\int_{\mathbb{R}^2}|\boldsymbol{t}|^{\alpha}|\check{f}(\boldsymbol{t})|^2 d\boldsymbol{t}. \tag{53}$$

By setting $(\xi_1,\xi_2) = (B_1\xi_1, B_2\xi_2)$, we obtain

$$\int_{\mathbb{R}^2}|B_1B_2||B_1B_2|^{-\alpha}|\boldsymbol{\xi}|^{-\alpha}|\mathcal{F}_H\{\check{f}\}(B_1\xi_1,B_2\xi_2)|^2 d\boldsymbol{\xi} \leq M(\alpha)\int_{\mathbb{R}^2}|\boldsymbol{t}|^{\alpha}|\check{f}(\boldsymbol{t})|^2 d\boldsymbol{t}. \tag{54}$$

By substituting Equations (37) and (39) into both sides of the above identity, the following result is obtained:

$$\begin{aligned}\int_{\mathbb{R}^2} |B_1B_2||B_1B_2|^{-\alpha}|\boldsymbol{\xi}|^{-\alpha}\left|e^{\mathbf{i}(C_1\xi_1^2+E_1\xi_1)}\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})e^{\mathbf{j}(C_2\xi_2^2+E_2\xi_2)}\right|^2 d\boldsymbol{\xi}\\ \leq M(\alpha)\int_{\mathbb{R}^2}|\boldsymbol{t}|^{\alpha}\left|e^{-\mathbf{i}D_1t_1}e^{-\mathbf{i}A_1t_1^2}f(\boldsymbol{t})e^{-\mathbf{j}A_2t_2^2}e^{-\mathbf{j}D_2t_2}\right|^2 d\boldsymbol{t}.\end{aligned} \tag{55}$$

Finally, we have

$$\int_{\mathbb{R}^2} |B_1B_2|^{1-\alpha}|\boldsymbol{\xi}|^{-\alpha}\left|\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} \leq M(\alpha)\int_{\mathbb{R}^2}|\boldsymbol{t}|^{\alpha}\left|f(\boldsymbol{t})\right|^2 d\boldsymbol{t}, \tag{56}$$

The proof is complete. □

Now, we are in a position to derive the Heisenberg-type uncertainty principle for the QQPFT, which is an extension of Theorem 1 mentioned earlier:

Theorem 10. *Under the same assumptions as in Theorem 1, one can obtain*

$$\int_{\mathbb{R}^2} t_k^2|f(\boldsymbol{t})|^2 d\boldsymbol{t}\int_{\mathbb{R}^2}\xi_k^2\left|\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} \geq \frac{1}{4B_k^2|B_1B_2|}\left(\int_{\mathbb{R}^2}|f(\boldsymbol{t})|^2 d\boldsymbol{t}\right)^2. \tag{57}$$

Proof. Replacing $f(\boldsymbol{t})$ with $\check{f}(\boldsymbol{t})$, described in Equation (37), into both sides of Equation (19) gives

$$\int_{\mathbb{R}^2} t_k^2|\check{f}(\boldsymbol{t})|^2 d\boldsymbol{t}\int_{\mathbb{R}^2}\xi_k^2\left|\mathcal{F}_H\{\check{f}\}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} \geq \frac{1}{4}\left(\int_{\mathbb{R}^2}|\check{f}(\boldsymbol{t})|^2 d\boldsymbol{t}\right)^2. \tag{58}$$

Consequently, we have

$$\int_{\mathbb{R}^2} t_k^2|\check{f}(\boldsymbol{t})|^2 d\boldsymbol{t}\int_{\mathbb{R}^2}|B_1B_2|B_k^2\xi_k^2\left|\mathcal{F}_H\{\check{f}\}(B_1\xi_1,B_2\xi_2)\right|^2 d\boldsymbol{\xi} \geq \frac{1}{4}\left(\int_{\mathbb{R}^2}|\check{f}(\boldsymbol{t})|^2 d\boldsymbol{t}\right)^2. \tag{59}$$

By inserting Equations (37) and (39) into the above identity, we find that

$$\begin{aligned}&\int_{\mathbb{R}^2} t_k^2\left|e^{-\mathbf{i}A_1t_1^2}e^{-\mathbf{i}D_1t_1}f(\boldsymbol{t})e^{-\mathbf{j}D_2t_2}e^{-\mathbf{j}A_2t_2^2}\right|^2 d\boldsymbol{t}\\ &\quad\times\int_{\mathbb{R}^2}B_k^2|B_1B_2|\xi_k^2\left|e^{\mathbf{i}(C_1\xi_1^2+E_1\xi_1)}\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})e^{\mathbf{j}(C_2\xi_2^2+E_2\xi_2)}\right|^2 d\boldsymbol{\xi}\\ &\quad\geq\frac{1}{4}\left(\int_{\mathbb{R}^2}\left|e^{-\mathbf{i}A_1t_1^2}e^{-\mathbf{i}D_1t_1}f(\boldsymbol{t})e^{-\mathbf{j}D_2t_2}e^{-\mathbf{j}A_2t_2^2}\right|^2 d\boldsymbol{t}\right)^2.\end{aligned} \tag{60}$$

Simplifying this gives the required result. □

In the following paragraph, we state and prove the logarithmic uncertainty principle related to the QQPFT:

Theorem 11. *Let $f \in \mathcal{S}(\mathbb{R}^2;\mathbb{H})$. Then, one obtains*

$$\begin{aligned}\int_{\mathbb{R}^2}\ln\sqrt{t_1^2+t_2^2}\left|f(\boldsymbol{t})\right|^2 d\boldsymbol{t}\int_{\mathbb{R}^2}\ln\sqrt{B_1^2\xi_1^2+B_2^2\xi_2^2}\left|\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi}\\ \geq |B_1B_2|^2\left[\frac{\Gamma'(1/2)}{\Gamma(1/2)}-\ln\pi\right]\int_{\mathbb{R}^2}\left|\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\right|^2 d\boldsymbol{t}.\end{aligned} \tag{61}$$

Proof. According to the logarithmic uncertainty principle concerning the QFT [27,28], we obtain

$$\int_{\mathbb{R}^2} \ln\sqrt{t_1^2+t_2^2}\,\left|\check{f}(\boldsymbol{t})\right|^2 d\boldsymbol{t} \int_{\mathbb{R}^2} \ln\sqrt{\xi_1^2+\xi_2^2}\,\left|\mathcal{F}_H\{\check{f}\}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} \geq \int_{\mathbb{R}^2}\left[\frac{\Gamma'(1/2)}{\Gamma(1/2)}-\ln\pi\right]|\check{f}(\boldsymbol{t})|^2\,d\boldsymbol{t}. \tag{62}$$

Upon putting $(B_1\xi_1, B_2\xi_2)$ in place of (ξ_1, ξ_2), we find

$$\int_{\mathbb{R}^2} \ln\sqrt{t_1^2+t_2^2}\,\left|\check{f}(\boldsymbol{t})\right|^2 d\boldsymbol{t} \int_{\mathbb{R}^2} \frac{1}{|B_1B_2|}\ln\sqrt{B_1^2\xi_1^2+B_2^2\xi_2^2}\,\left|\mathcal{F}_H\{\check{f}\}(B_1\xi_1,B_2\xi_2)\right|^2 d\boldsymbol{\xi} \geq \int_{\mathbb{R}^2}\left[\frac{\Gamma'(1/2)}{\Gamma(1/2)}-\ln\pi\right]|\check{f}(\boldsymbol{t})|^2\,d\boldsymbol{t}. \tag{63}$$

Through Equation (39), we infer that

$$\int_{\mathbb{R}^2} \ln\sqrt{t_1^2+t_2^2}|\check{f}(\boldsymbol{t})|^2 d\boldsymbol{t} \int_{\mathbb{R}^2} \frac{1}{|B_1B_2|}\ln\sqrt{B_1^2\xi_1^2+B_2^2\xi_2^2} \times\left|e^{\mathbf{i}(C_1\xi_1^2+E_1\xi_1)}\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})e^{\mathbf{j}(C_2\xi_2^2+E_2\xi_2)}\right|^2 d\boldsymbol{\xi} \geq \int_{\mathbb{R}^2}\left[\frac{\Gamma'(1/2)}{\Gamma(1/2)}-\ln\pi\right]|\check{f}(\boldsymbol{t})|^2\,d\boldsymbol{t}. \tag{64}$$

By simplifying and using Equation (38), we see that

$$\int_{\mathbb{R}^2} \ln\sqrt{t_1^2+t_2^2}\,|f(\boldsymbol{t})|^2 d\boldsymbol{t} \int_{\mathbb{R}^2} \frac{1}{|B_1B_2|}\ln\sqrt{B_1^2\xi_1^2+B_2^2\xi_2^2}\left|\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} \geq \int_{\mathbb{R}^2}\left[\frac{\Gamma'(1/2)}{\Gamma(1/2)}-\ln\pi\right]|f(\boldsymbol{t})|^2\,d\boldsymbol{t}. \tag{65}$$

Through Equation (45), we obtain

$$\int_{\mathbb{R}^2} \ln\sqrt{t_1^2+t_2^2}\,|f(\boldsymbol{t})|^2 d\boldsymbol{t} \int_{\mathbb{R}^2} \frac{1}{|B_1B_2|}\ln\sqrt{B_1^2\xi_1^2+B_2^2\xi_2^2}\left|\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} \geq \int_{\mathbb{R}^2}\left[\frac{\Gamma'(1/2)}{\Gamma(1/2)}-\ln\pi\right]|B_1B_2||\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})|^2\,d\boldsymbol{t},$$

which gives the desired result. □

Observe that for $1 \leq p \leq 2$, we may extend the L^2-norm to the L^p-norm in Equation (57) and obtain the following result:

Theorem 12. *Under the conditions in Theorem 1, the following holds:*

$$\int_{\mathbb{R}^2} t_k^p|f(\boldsymbol{t})|^p\,d\boldsymbol{t}\int_{\mathbb{R}^2}\xi_k^p|\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})|^p\,d\boldsymbol{\xi} \geq \frac{1}{2^p}|B_1B_2|^{-1}|B_k|^{-p}\left(\int_{\mathbb{R}^2}|f(\boldsymbol{t})|^2\,d\boldsymbol{t}\right)^p, \quad k=1,2. \tag{66}$$

In particular, we have

$$\int_{\mathbb{R}^2} t_k^p|f(\boldsymbol{t})|^p\,d\boldsymbol{t}\int_{\mathbb{R}^2}\xi_k^p|\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})|^p\,d\boldsymbol{\xi} \geq \frac{1}{2^p}|B_k|^{-p}\left(\int_{\mathbb{R}^2}|\mathcal{Q}_{Q_1,Q_2}\{f\}|^2\,d\boldsymbol{t}\right)^{1/p}. \tag{67}$$

for which $\frac{1}{p}+\frac{1}{q}=1$.

Proof. In fact, we have

$$\left(\int_{\mathbb{R}^2} t_k^p |f(\boldsymbol{t})|^p \, d\boldsymbol{\xi}\right)^{1/p} \left(\int_{\mathbb{R}^2} \xi_k^p |\mathcal{F}_H\{f\}(\boldsymbol{\xi})|^p \, d\boldsymbol{\xi}\right)^{1/p} \geq \frac{1}{2} \int_{\mathbb{R}^2} |f(\boldsymbol{t})|^2 \, d\boldsymbol{t}. \tag{68}$$

Replacing $f(\boldsymbol{t})$ with $\check{f}(\boldsymbol{t})$, described in Equation (37), into both sides of Equation (68) yields

$$\left(\int_{\mathbb{R}^2} t_k^p |\check{f}(\boldsymbol{t})|^p \, d\boldsymbol{t}\right)^{1/p} \left(\int_{\mathbb{R}^2} \xi_k^p |\mathcal{F}_H\{\check{f}\}(\boldsymbol{\xi})|^p \, d\boldsymbol{\xi}\right)^{1/p} \geq \frac{1}{2} \int_{\mathbb{R}^2} |\check{f}(\boldsymbol{t})|^2 \, d\boldsymbol{t}. \tag{69}$$

Hence, we have

$$\left(\int_{\mathbb{R}^2} t_k^p |\check{f}(\boldsymbol{t})|^p \, d\boldsymbol{t}\right)^{1/p} \left(\int_{\mathbb{R}^2} |B_1B_2||B_k|^p \xi_k^p |\mathcal{F}_H\{\check{f}\}(B_1\xi_1, B_2\xi_2)|^p \, d\boldsymbol{\xi}\right)^{1/p} \geq \frac{1}{2} \int_{\mathbb{R}^2} |\check{f}(\boldsymbol{t})|^2 \, d\boldsymbol{t}. \tag{70}$$

Through the application of Equations (38)–(40) into the above identity, we obtain

$$\left(\int_{\mathbb{R}^2} t_k^p |f(\boldsymbol{t})|^p \, d\boldsymbol{t}\right)^{1/p} \left(\int_{\mathbb{R}^2} |B_1B_2||B_k|^p \xi_k^p |\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})|^p \, d\boldsymbol{\xi}\right)^{1/p} \geq \frac{1}{2} \int_{\mathbb{R}^2} |f(\boldsymbol{t})|^2 \, d\boldsymbol{t}. \tag{71}$$

This is equivalent to

$$\int_{\mathbb{R}^2} t_k^p |f(\boldsymbol{t})|^p \, d\boldsymbol{t} \int_{\mathbb{R}^2} \xi_k^p |\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})|^p \, d\boldsymbol{\xi} \geq \frac{1}{2^p} |B_1B_2|^{-1} |B_k|^{-p} \left(\int_{\mathbb{R}^2} |f(\boldsymbol{t})|^2 \, d\boldsymbol{t}\right)^p.$$

Under Equation (45), we obtain

$$\int_{\mathbb{R}^2} t_k^p |f(\boldsymbol{t})|^p \, d\boldsymbol{t} \int_{\mathbb{R}^2} \xi_k^p |\mathcal{Q}_{Q_1,Q_2}\{f\}(\boldsymbol{\xi})|^p \, d\boldsymbol{\xi} \geq \frac{1}{2^p} |B_k|^{-p} \left(\int_{\mathbb{R}^2} |\mathcal{Q}_{Q_1,Q_2}\{f\}|^2 \, d\boldsymbol{t}\right)^{1/p}.$$

This ends the proof of the theorem. □

Remark 2. *It should be observed that for $p = 2$, Theorem 12 changes to Theorem 10.*

6. Conclusions

In this paper, we introduced the quaternion quadratic-phase Fourier transform and investigated properties such as the continuous and Riemann–Lebesgue properties. We derived several inequalities related to this transformation. The proof of this uncertainty utilizes the basic relationship between the quaternion quadratic-phase Fourier transform and quaternion Fourier transform. Some possible future extensions include the application of the proposed method in image processing, which is well documented in [29,30], as well as in computer graphics, scientific visualization and numerical analysis, which have been explored in [31–34].

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