mathematics

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Fractional Fourier Transform: Main Properties and Inequalities

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Abstract: The fractional Fourier transform is a natural generalization of the Fourier transform. In this work, we recall the definition of the fractional Fourier transform and its relation to the conventional Fourier transform. We exhibit that this relation permits one to obtain easily the main properties of the fractional Fourier transform. We investigate the sharp Hausdorff-Young inequality for the fractional Fourier transform and utilize it to build Matolcsi-Szücs inequality related to this transform. The other versions of the inequalities concerning the fractional Fourier transform is also discussed in detail. The results obtained in this paper are very significant, especially in the field of fractional differential equations.

Keywords: fractional Fourier transform; uncertainty principle; Donoho-Stark uncertainty principle

MSC: 11R52; 42A38; 15A66; 83A05; 35L05

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Citation: Bahri, M.; Abdul Karim, S.A. Fractional Fourier Transform: Main Properties and Inequalities. *Mathematics* **2023**, *11*, 1234. https://doi.org/10.3390/math11051234

Academic Editor: Luigi Rodino

Received: 24 December 2022
Revised: 7 February 2023
Accepted: 7 February 2023
Published: 3 March 2023

1. Introduction

The fractional Fourier transform, which was originally introduced by [1], has received considerable attention in recent years from both theoretical and practical points of view. This important transform is also known as a nontrivial generalization of the conventional Fourier transform (FT). Researchers in [2–7] have focused on investigating the fundamental properties of the fractional Fourier transform such as convolution, correlation and inequalities. These properties are expansions of well known results of the FT and other related transforms (see, e.g., [8–11]. The authors of [12–15]) have developed its applications in areas like optics, filter design, pattern recognition. Some researchers also have been interested in generalizing various transformations utilizing the kernel of the fractional Fourier transform. For instance, the authors in [16–18] have proposed an extension of Stockwell transform so-called the fractional Stockwell transform. Several properties of the transform were also derived in some detail. In [19–22], the authors have successfully presented the fractional wavelet transform, which is a generalization of the conventional wavelet transform [23–25] in the fractional Fourier domain. They found that the properties of new wavelet transform are a modification of the corresponding wavelet transform properties. Keeping in view the fact that the uncertainty principles for the fractional Fourier transforms are one of the most fundamental results associated with this transform. In [23,26–33], the authors have demonstrated many variants and generalizations of the uncertainty principle for various transformations.

In the present work, we first introduce the definition of the fractional Fourier transform (FRFT) and basic properties. As was discussed in [34,35], we propose another version of a natural link between the fractional Fourier transform and Fourier transform. We show that the Heisenberg-type uncertainty principles for the fractional Fourier transforms can be obtained using the interesting relation. We also establish the other versions of the uncertainty principles in framework of the fractional Fourier transform like Matolcsi-Szücs inequality, Donoho-Stark uncertainty principles, and Shannon entropy uncertainty principle. It is emphasized that the proposed uncertainty principles in this work are quite different from those that have been studied in [3–5].

We display here the plan of this paper. In Section 2 we remind definition of the fractional Fourier transform and basic notations that will be useful later. Also, in this section the basic relationship between the Fourier transform and fractional Fourier transform are studied. Section 3 presents three Heisenberg-type uncertainty principles for the fractional Fourier transform, which are generalizations of the famous uncertainty principles related to the Fourier transform. We extend Matolcsi-Szücs inequality to the fractional Fourier transform in Section 4. In Section 5 we focus our attention to the derivation of Donoho-Stark uncertainty principle associated with the fractional Fourier transform. In Section 6 we generalize Shannon entropy in the framework of the fractional Fourier transform. A conclusion may be found at the end of this work.

2. Fractional Fourier Transform

First of all we discuss the relevant materials related to the fractional Fourier transform (FRFT) [1,13,36], including the basic connection between the Fourier transform and fractional Fourier transform, which will useful in the sequel. We first introduce some basic symbols shown in Table 1.

Table 1. Symbol Description.

Symbols	Description
$\mathcal{F}$	Fourier transform
$\mathcal{F}_\theta$	Fractional Fouirer transform
Pr	Projection operator
E, T	Measurable sets
$\mathbb{R}$	Real Numbers

Definition 1. *Fix $1 \leq r < \infty$, for measurable functions on $\mathbb{R}$ we define the linear space $L^r(\mathbb{R})$-norm as*

$$\|f\|_{L^r(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(t)|^r \, dt \right)^{1/r} < \infty. \tag{1}$$

Clearly for $r \to \infty$ we obtain $L^\infty(\mathbb{R})$-norm

$$\|f\|_{L^\infty(\mathbb{R})} = \text{ess sup}_{t\in\mathbb{R}} |f(t)|. \tag{2}$$

If $f \in L^\infty(\mathbb{R})$ is continuous then

$$\|f\|_{L^\infty(\mathbb{R})} = \sup_{t\in\mathbb{R}} |f(t)|. \tag{3}$$

The inner product of $L^2(\mathbb{R})$ is defined as

$$\langle f, g\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(t)\overline{g(t)} \, dt. \tag{4}$$

We remind that the Fourier transform of a function $f \in L^2(\mathbb{R})$ is given by (see [37,38])

$$\mathcal{F}\{f\}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\xi) e^{-ix\xi} \, dx. \tag{5}$$

Definition 2. *For a given function $f \in L^1(\mathbb{R})$, its FRFT with parameter θ is given by the integral*

$$\mathcal{F}_\theta\{f\}(\xi) = \int_{\mathbb{R}} f(t) K_\theta(\xi, t)\, dt, \tag{6}$$

where the kernel transform $K_\theta(\xi, t)$ is defined by

$$K_\theta(\xi, t) = \begin{cases} A_\theta e^{i(t^2+\xi^2)\frac{\cot\theta}{2} - it\xi\csc\theta}, & \theta \neq n\pi \\ \delta(t-\xi), & \theta = 2n\pi \\ \delta(t+\xi), & \theta = (2n+1)\pi, n \in \mathbb{Z}. \end{cases} \tag{7}$$

Here δ is a Dirac delta function and

$$A_\theta = \frac{e^{i(\frac{\theta}{2} - \frac{\pi}{4})}}{\sqrt{2\pi \sin\theta}} = \sqrt{\frac{1 - i\cot\theta}{2\pi}}, \quad \overline{A_\theta} = \sqrt{\frac{1 + i\cot\theta}{2\pi}} \tag{8}$$

It is evident that

$$|A_\theta| = \sqrt{\frac{\csc\theta}{2\pi}}, \quad \csc\theta = \frac{1}{\sin\theta}. \tag{9}$$

It is straightforward to check that the FRFT kernel satisfies the following important properties:

$$\overline{K_\theta(\xi, t)} = K_{-\theta}(\xi, t),$$

and

$$\int_{\mathbb{R}} K_\theta(\xi, t) \overline{K_\theta(\xi', t)}\, dt = \delta(\xi - \xi'),$$

where $\overline{K_\theta(\xi, t)}$ is the complex conjugate of $K_\theta(\xi, t)$.

Lemma 1. *For every $f \in L^1(\mathbb{R})$ and $\mathcal{F}_\theta\{f\} \in L^\infty(\mathbb{R})$ we have*

$$\|\mathcal{F}_\theta\{f\}\|_{L^\infty(\mathbb{R})} \leq |A_\theta| \|f\|_{L^1(\mathbb{R})}. \tag{10}$$

Proof. It directly follows with (3) and (6) that

$$|\mathcal{F}_\theta\{f\}(\xi)| \leq |A_\theta| \int_{\mathbb{R}} \left| f(t) e^{i(t^2+\xi^2)\frac{\cot\theta}{2} - it\xi\csc\theta} \right| dt. \tag{11}$$

This clearly implies that

$$\begin{aligned} \|\mathcal{F}_\theta\{f\}\|_{L^\infty(\mathbb{R})} &= \sup_{\xi \in \mathbb{R}} |\mathcal{F}_\theta\{f\}(\xi)| \\ &\leq |A_\theta| \int_{\mathbb{R}} |f(t)|\, dt \\ &= |A_\theta| \|f\|_{L^1(\mathbb{R})}. \end{aligned}$$

This is required result. □

Definition 3. *Suppose that $f \in L^1(\mathbb{R})$ and $\mathcal{F}_\theta\{f\} \in L^1(\mathbb{R})$. The inverse FRFT of f is given by the integral*

$$\begin{aligned} f(t) &= \mathcal{F}_\theta^{-1}(\mathcal{F}_\theta\{f\}) \\ &= \int_{\mathbb{R}} \mathcal{F}_\theta\{f\}(\xi) K_{-\theta}(\xi, t)\, d\xi \\ &= \int_{\mathbb{R}} \mathcal{F}_\theta\{f\}(\xi) \overline{A_\theta} e^{-i(t^2+\xi^2)\frac{\cot\theta}{2} + it\xi\csc\theta}\, d\xi. \end{aligned} \tag{12}$$

Below we express a fundamental connection between the Fourier transform and fractional Fourier transform. Due to the FRFT definition (6) we have

$$\begin{aligned} \mathcal{F}_\theta\{f\}(\xi) &= A_\theta \int_{\mathbb{R}} f(t) e^{i(t^2+\xi^2)\frac{\cot\theta}{2} - it\xi\csc\theta}\, dt \\ &= A_\theta e^{i\xi^2\frac{\cot\theta}{2}} \int_{\mathbb{R}} f(t) e^{it^2\frac{\cot\theta}{2} - it\xi\csc\theta}\, dt \\ &= A_\theta e^{i\xi^2\frac{\cot\theta}{2}} \sqrt{2\pi}\mathcal{F}\{e^{it^2\frac{\cot\theta}{2}} f(t)\}(\xi\csc\theta). \end{aligned} \tag{13}$$

Denoted by

$$f_\theta(t) = e^{it^2\frac{\cot\theta}{2}} f(t), \tag{14}$$

then we obtain

$$\frac{e^{-i\xi^2\frac{\cot\theta}{2}}}{\sqrt{1-i\cot\theta}} \mathcal{F}_\theta\{f\}(\xi) = \mathcal{F}\{f_\theta\}(\xi\csc\theta). \tag{15}$$

Hence,

$$\frac{e^{-i\xi^2\frac{\cot\theta}{2}}}{\sqrt{1-i\cot\theta}} \mathcal{F}_\theta\{e^{it^2\frac{\cot\theta}{2}} f\}(\xi) = \mathcal{F}\{f\}\big(\xi\csc\theta\big). \tag{16}$$

Now we provide the different proof of Parseval formula for the FRFT using the direct relationship between the FT and FRFT.

Lemma 2 (FRFT Parseval)**.** *For all $f, g \in L^2(\mathbb{R})$, the following relation holds:*

$$\langle f, g\rangle_{L^2(\mathbb{R})} = \langle \mathcal{F}_\theta\{f\}, \mathcal{F}_\theta\{g\}\rangle_{L^2(\mathbb{R})}, \tag{17}$$

and

$$\|f\|^2_{L^2(\mathbb{R})} = \|\mathcal{F}_\theta\{f\}\|^2_{L^2(\mathbb{R})}. \tag{18}$$

Proof. According to the Parseval's formula for the FT we immediately obtain

$$\int_{\mathbb{R}} f(t)\overline{g(t)}\, dt = \int_{\mathbb{R}} \mathcal{F}\{f\}(\xi)\overline{\mathcal{F}\{g\}(\xi)}\, d\xi. \tag{19}$$

Replacement of f by f_θ and g by g_θ on both sides of (19) we see that

$$\begin{aligned} \int_{\mathbb{R}} f_\theta(t)\overline{g_\theta(t)}\, dt &= \int_{\mathbb{R}} \mathcal{F}\{f_\theta\}(\xi)\overline{\mathcal{F}\{g_\theta\}(\xi)}\, d\xi \\ \int_{\mathbb{R}} e^{it^2\frac{\cot\theta}{2}} f(t)\, \overline{e^{it^2\frac{\cot\theta}{2}} g(t)}\, dt &= \int_{\mathbb{R}} \mathcal{F}\{f_\theta\}(\xi)\overline{\mathcal{F}\{g_\theta\}(\xi)}\, d\xi. \end{aligned} \tag{20}$$

An application of relations (13) and (14) will lead to

$$\begin{aligned} \int_{\mathbb{R}} f(t)\overline{g(t)}\, dt &= \int_{\mathbb{R}} \mathcal{F}\{f_\theta\}(\csc\theta\xi)\overline{\mathcal{F}\{g_\theta\}(\csc\theta\xi)}\, d\csc\theta\xi \\ &= |\csc\theta| \int_{\mathbb{R}} \frac{e^{-i\xi^2\frac{\cot\theta}{2}}}{\sqrt{1-i\cot\theta}} \mathcal{F}_\theta\{f\}(\xi)\, \overline{\frac{e^{-i\xi^2\frac{\cot\theta}{2}}}{\sqrt{1-i\cot\theta}} \mathcal{F}_\theta\{g\}(\xi)}\, d\xi \\ &= \frac{|\csc\theta|}{|1-i\cot\theta|} \int_{\mathbb{R}} \mathcal{F}_\theta\{f\}(\xi)\overline{\mathcal{F}_\theta\{g\}(\xi)}\, d\xi \\ &= \int_{\mathbb{R}} \mathcal{F}_\theta\{f\}(\xi)\overline{\mathcal{F}_\theta\{g\}(\xi)}\, d\xi, \end{aligned} \tag{21}$$

which proves the theorem. □

3. Heisenberg-Type Uncertainty Principles for FRFT

Our interest is to derive three Heisenberg-type uncertainty principles involving the FRFT. They are generalized forms of the Heisenberg-type uncertainty principles related to the conventional Fourier transform.

Heisenberg-type uncertainty principles for the FRFT explains the function $f(t)$ is the probability that a particle's position is ξ, and its fractional Fourier transform $\mathcal{F}_\theta\{f\}$ is the probability that its momentum is ξ, then the principle informs a lower bound on how spread out these two probability distributions must be.

Theorem 1. *Let f be in $L^2(\mathbb{R})$. Then the following inequality holds:*

$$\int_{\mathbb{R}} |t|^2 |f(t)|^2\, dt \int_{\mathbb{R}} |\xi|^2 |\mathcal{F}_\theta\{f\}(\xi)|^2\, d\xi \geq \frac{|\sin\theta|^2}{4}\left(\int_{\mathbb{R}} |f(t)|^2\, dt\right)^2. \tag{22}$$

Proof. By virtue of the uncertainty principle for the FT we obtain

$$\int_{\mathbb{R}} |t|^2 |f_\theta(t)|^2\, dt \int_{\mathbb{R}} |\xi|^2 |\mathcal{F}_\theta\{f_\theta\}(\xi)|^2\, d\xi \geq \frac{1}{4}\left(\int_{\mathbb{R}} |f_\theta(t)|^2\, dt\right)^2. \tag{23}$$

We have from (14)

$$\int_{\mathbb{R}} |t|^2 |e^{it^2\frac{\cot\theta}{2}} f(t)|^2\, dt \int_{\mathbb{R}} |\xi\csc\theta|^2 |\mathcal{F}\{f_\theta\}(\xi\csc\theta)|^2\, d(\xi\csc\theta) \geq \frac{1}{4}\left(\int_{\mathbb{R}} |e^{it^2\frac{\cot\theta}{2}} f(t)|^2\, dt\right)^2$$

$$\int_{\mathbb{R}} |t|^2 |f(t)|^2\, dt \int_{\mathbb{R}} |\xi|^2 (\csc\theta)^3 |\mathcal{F}\{f_\theta\}(\xi\csc\theta)|^p\, d\xi \geq \frac{1}{4}\left(\int_{\mathbb{R}} |f(t)|^2\, dt\right)^2$$

$$|\csc\theta|^3 \int_{\mathbb{R}} |t|^2 |f(t)|^2\, dt \int_{\mathbb{R}} |\xi|^2 |\mathcal{F}\{f_\theta\}(\xi\csc\theta)|^2\, d\xi \geq \frac{1}{4}\left(\int_{\mathbb{R}} |f(t)|^p\, dt\right)^2.$$

Applying (15) yields

$$|\csc\theta|^3 \int_{\mathbb{R}} |t|^2 |f(t)|^2\, dt \int_{\mathbb{R}} |\xi|^2 \left|\frac{e^{-i\xi^2\frac{\cot\theta}{2}}}{\sqrt{1-i\cot\theta}}\mathcal{F}_\theta\{f\}(\xi)\right|^2 d\xi \geq \frac{1}{4}\left(\int_{\mathbb{R}} |f(t)|^2\, dt\right)^2$$

$$\frac{|\csc\theta|^3}{|1-i\cot\theta|} \int_{\mathbb{R}} |t|^2 |f(t)|^2\, dt \int_{\mathbb{R}} |\xi|^2 |\mathcal{F}_\theta\{f\}(\xi)|^2\, d\xi \geq \frac{1}{4}\left(\int_{\mathbb{R}} |f(t)|^2\, dt\right)^2$$

$$|\csc\theta|^3 |\sin\theta| \int_{\mathbb{R}} |t|^2 |f(t)|^2\, dt \int_{\mathbb{R}} |\xi|^2 |\mathcal{F}_\theta\{f\}(\xi)|^2\, d\xi \geq \frac{1}{4}\left(\int_{\mathbb{R}} |f(t)|^2\, dt\right)^2$$

$$\int_{\mathbb{R}} |t|^2 |f(t)|^2\, dt \int_{\mathbb{R}} |\xi|^2 |\mathcal{F}_\theta\{f\}(\xi)|^2\, d\xi \geq \frac{|\sin\theta|^2}{4}\left(\int_{\mathbb{R}} |f(t)|^2\, dt\right)^2,$$

and the proof is complete. □

A simple modification of Theorem 1 yields the following variant.

Theorem 2. *For any $f \in L^2(\mathbb{R})$ and $\alpha, \gamma \geq 1$, one has*

$$\left(\int_{\mathbb{R}} |t|^{2\gamma} |f(t)|^2\, dt\right)^{\frac{\alpha}{\alpha+\gamma}} \left(\int_{\mathbb{R}} |\xi|^{2\alpha} |\mathcal{F}_\theta\{f\}(\xi)|^2\, d\xi\right)^{\frac{\gamma}{\alpha+\gamma}} \geq \left(\frac{|\sin\theta|^2}{4}\right)^{\frac{\alpha\gamma}{\alpha+\gamma}} \|f\|^2_{L^2(\mathbb{R})}. \tag{24}$$

Proof. It follows from Hölder's inequality and Plancherel theorem for the FRFT (18) that

$$\begin{aligned}
&\int_{\mathbb{R}} |\xi|^2 \left|\mathcal{F}_\theta\{f\}(\xi)\right|^2 d\xi \\
&\quad = \int_{\mathbb{R}} |\xi|^2 \left|\mathcal{F}_\theta\{f\}(\xi)\right|^{\frac{2}{\alpha}} \left|\mathcal{F}_\theta\{f\}(\xi)\right|^{2-\frac{2}{\alpha}} d\xi \\
&\quad \leq \left(\int_{\mathbb{R}} \left(|\xi|^2 \left|\mathcal{F}_\theta\{f\}(\xi)\right|^{\frac{2}{\alpha}} \right)^{\alpha} d\xi \right)^{\frac{1}{\alpha}} \left(\int_{\mathbb{R}} \left(\left|\mathcal{F}_\theta\{f\}(\xi)\right|^{2-\frac{2}{\alpha}} \right)^{\frac{\alpha}{\alpha-1}} d\xi \right)^{\frac{\alpha-1}{\alpha}} \\
&\quad = \left(\int_{\mathbb{R}} |\xi|^{2\alpha} \left|\mathcal{F}_\theta\{f\}(\xi)\right|^2 d\xi \right)^{\frac{1}{\alpha}} \left(\int_{\mathbb{R}} \left|\mathcal{F}_\theta\{f\}(\xi)\right|^2 d\xi \right)^{1-\frac{1}{\alpha}} \\
&\quad = \left(\int_{\mathbb{R}} |\xi|^{2\alpha} \left|\mathcal{F}_\theta\{f\}(\xi)\right|^2 d\xi \right)^{\frac{1}{\alpha}} \left(\int_{\mathbb{R}} |f(t)|^2 dt \right)^{1-\frac{1}{\alpha}}.
\end{aligned} \tag{25}$$

Hence,

$$\left(\int_{\mathbb{R}} |\xi|^{2\alpha} \left|\mathcal{F}_\theta\{f\}(\xi)\right|^2 d\xi \right)^{\frac{1}{\alpha}} \geq \frac{\int_{\mathbb{R}} |\xi|^2 \left|\mathcal{F}_\theta\{f\}(\xi)\right|^2 d\xi}{\|f\|_{L^2(\mathbb{R})}^{2(\frac{\alpha-1}{\alpha})}}. \tag{26}$$

In a similar way, we get

$$\left(\int_{\mathbb{R}} |t|^{2\gamma} |f(t)|^2 dt \right)^{\frac{1}{\gamma}} \geq \frac{\int_{\mathbb{R}} |t|^2 |f(t)|^2 dt}{\|f\|_{L^2(\mathbb{R})}^{2(\frac{\gamma-1}{\gamma})}}. \tag{27}$$

By combining (26) and (27) we obtain

$$\left(\int_{\mathbb{R}} |t|^{2\gamma} |f(t)|^2 dt \right)^{\frac{1}{\gamma}} \left(\int_{\mathbb{R}} |\xi|^{2\alpha} \left|\mathcal{F}_\theta\{f\}(\xi)\right|^2 d\xi \right)^{\frac{1}{\alpha}} \geq \frac{\int_{\mathbb{R}} |t|^2 |f(t)|^2 dt \int_{\mathbb{R}} |\xi|^2 \left|\mathcal{F}_\theta\{f\}(\xi)\right.}{\|f\|_{L^2(\mathbb{R})}^{2(\frac{\gamma-1}{\gamma}+\frac{\alpha-1}{\alpha})}}. \tag{28}$$

Applying (22) in Theorem 1 results in

$$\left(\int_{\mathbb{R}} |t|^{2\gamma} |f(t)|^2 dt \right)^{\frac{1}{\gamma}} \left(\int_{\mathbb{R}} |\xi|^{2\alpha} \left|\mathcal{F}_\theta\{f\}(\xi)\right|^2 d\xi \right)^{\frac{1}{\alpha}} \geq \frac{|\sin\theta|^2}{4} \|f\|_{L^2(\mathbb{R})}^{2(\frac{\alpha+\gamma}{\alpha\gamma})}. \tag{29}$$

Or, equivalently,

$$\left(\int_{\mathbb{R}} |t|^{2\gamma} |f(t)|^2 dt \right)^{\frac{\alpha}{\alpha+\gamma}} \left(\int_{\mathbb{R}} |\xi|^{2\alpha} \left|\mathcal{F}_\theta\{f\}(\xi)\right|^2 d\xi \right)^{\frac{\gamma}{\alpha+\gamma}} \geq \left(\frac{|\sin\theta|^2}{4} \right)^{\frac{\alpha\gamma}{\alpha+\gamma}} \|f\|_{L^2(\mathbb{R})}^2, \tag{30}$$

and the proof is complete. □

The extension of Theorem 1 is also showed by the following result.

Theorem 3. *Under the same conditions as in Theorem 1 we have*

$$\int_{\mathbb{R}} |t|^s |f(t)|^s dt \int_{\mathbb{R}} |\xi|^s \left|\mathcal{F}_\theta\{f\}(\xi)\right|^s d\xi \geq \frac{|\sin\theta|^{\frac{s}{2}+1}}{2^s} \left(\int_{\mathbb{R}} |f(t)|^2 dt \right)^s \tag{31}$$

for $1 \leq s \leq 2$.

Proof. It follows by means of the Heisenberg's uncertainty for the conventional Fourier transform that

$$\int_{\mathbb{R}} |t|^s |f(t)|^s \, dt \int_{\mathbb{R}} |\xi|^s |\mathcal{F}\{f\}(\xi)|^s \, d\xi \geq \frac{1}{2^s} \left(\int_{\mathbb{R}} |f(t)|^2 \, dt \right)^s.$$

Since f_θ defined by (14) is in $L^2(\mathbb{R})$, then by replacing f by f_θ into the above identity, we obtain

$$\int_{\mathbb{R}} |t|^s |f_\theta(t)|^s \, dt \int_{\mathbb{R}} |\xi|^s |\mathcal{F}\{f_\theta\}(\xi)|^s \, d\xi \geq \frac{1}{2^s} \left(\int_{\mathbb{R}} |f_\theta(t)|^2 \, dt \right)^s.$$

Putting $\xi = \xi \csc\theta$ and applying (14) we get

$$\begin{aligned}
\int_{\mathbb{R}} |t|^s \left| e^{it^2 \frac{\cot\theta}{2}} f(t) \right|^s dt \int_{\mathbb{R}} |\xi \csc\theta|^s \left| \mathcal{F}\{f_\theta\}(\xi \csc\theta) \right|^s d(\xi \csc\theta) &\geq \frac{1}{2^s} \left(\int_{\mathbb{R}} \left| e^{it^2 \frac{\cot\theta}{2}} f(t) \right|^2 dt \right)^s \\
\int_{\mathbb{R}} |t|^s \left| f(t) \right|^s dt \int_{\mathbb{R}} |\xi|^s (\csc\theta)^{s+1} \left| \mathcal{F}\{f_\theta\}(\xi \csc\theta) \right|^s d\xi &\geq \frac{1}{2^s} \left(\int_{\mathbb{R}} \left| f(t) \right|^s dt \right)^s \\
|\csc\theta|^{s+1} \int_{\mathbb{R}} t^s \left| f(t) \right|^s dt \int_{\mathbb{R}} |\xi|^s \left| \mathcal{F}\{f_\theta\}(\xi \csc\theta) \right|^s d\xi &\geq \frac{1}{2^s} \left(\int_{\mathbb{R}} \left| f(t) \right|^2 dt \right)^s
\end{aligned}$$

Due to (15) we have

$$\begin{aligned}
|\csc\theta|^{s+1} \int_{\mathbb{R}} t^s \left| f(t) \right|^s dt \int_{\mathbb{R}} |\xi|^s \left| \frac{e^{-i\xi^2 \frac{\cot\theta}{2}}}{\sqrt{1 - i\cot\theta}} \mathcal{F}_\theta\{f\}(\xi) \right|^s d\xi &\geq \frac{1}{2^s} \left(\int_{\mathbb{R}} \left| f(t) \right|^2 dt \right)^s \\
\frac{|\csc\theta|^{s+1}}{|1 - i\cot\theta|^{\frac{s}{2}}} \int_{\mathbb{R}} |t|^s \left| f(t) \right|^s dt \int_{\mathbb{R}} |\xi|^s \left| \mathcal{F}_\theta\{f\}(\xi) \right|^s d\xi &\geq \frac{1}{2^s} \left(\int_{\mathbb{R}} \left| f(t) \right|^2 dt \right)^s \\
|\csc\theta|^{s+1} |\sin\theta|^{\frac{s}{2}} \int_{\mathbb{R}} |t|^s \left| f(t) \right|^s dt \int_{\mathbb{R}} |\xi|^s \left| \mathcal{F}_\theta\{f\}(\xi) \right|^s d\xi &\geq \frac{1}{2^s} \left(\int_{\mathbb{R}} \left| f(t) \right|^2 dt \right)^s \\
\int_{\mathbb{R}} |t|^s \left| f(t) \right|^s dt \int_{\mathbb{R}} |\xi|^s \left| \mathcal{F}_\theta\{f\}(\xi) \right|^s d\xi &\geq \frac{|\sin\theta|^{\frac{s}{2}+1}}{2^s} \left(\int_{\mathbb{R}} \left| f(t) \right|^2 dt \right)^s,
\end{aligned}$$

and the proof is complete. □

4. Matolcsi-Szücs Uncertainty Principles

We first state the following result, which describes the sharp Hausdorff-Young inequality related to the FRFT.

Theorem 4 (FRFT Hausdorff-Young)**.** *For any $1 \leq r \leq 2$ such that $\frac{1}{r} + \frac{1}{s} = 1$. Then for every function f in $L^r(\mathbb{R})$, it holds*

$$\left(\int_{\mathbb{R}} \left| \mathcal{F}_\theta\{f\}(\xi) \right|^s d\xi \right)^{1/s} \leq |\sin\theta|^{\frac{1}{s} - \frac{1}{2}} r^{1/2r} s^{-1/2s} \left(\int_{\mathbb{R}} |f(t)|^r \, dt \right)^{1/r}. \tag{32}$$

Proof. Applying the sharp Hausdorff-Young inequality related to the conventional Fourier transform results in

$$\left(\int_{\mathbb{R}} |\mathcal{F}\{f\}(\xi)|^s \, d\xi \right)^{1/s} \leq r^{1/2r} s^{-1/2s} \left(\int_{\mathbb{R}} |f(t)|^r \, dt \right)^{1/r}. \tag{33}$$

Including f_θ defined by (14) into both sides of (33) yields

$$\left(\int_{\mathbb{R}} |\mathcal{F}\{f_\theta\}(\xi)|^s \, d\xi \right)^{1/s} \leq r^{1/2r} s^{-1/2s} \left(\int_{\mathbb{R}} |f_\theta(t)|^r \, dt \right)^{1/r}. \tag{34}$$

We further have

$$\left(\int_{\mathbb{R}} |\mathcal{F}\{f_\theta\}(\xi \csc\theta)|^s \, d\xi \csc\theta\right)^{1/s} \leq r^{1/2r} s^{-1/2s} \left(\int_{\mathbb{R}} |f_\theta(t)|^r \, dt\right)^{1/r}. \tag{35}$$

Using (15) we may write the above identity as

$$\left(\int_{\mathbb{R}} \left| \frac{e^{-i\xi^2 \frac{\cot\theta}{2}}}{\sqrt{1 - i\cot\theta}} \mathcal{F}_\theta\{f\}(\xi)\right|^s \, d\xi \csc\theta\right)^{1/s} \leq r^{1/2r} s^{-1/2s} \left(\int_{\mathbb{R}} |e^{it^2 \frac{\cot\theta}{2}} f(t)|^r \, dt\right)^{1/r}. \tag{36}$$

Thus

$$\left(|\sin\theta|^{\frac{s}{2}-1} \int_{\mathbb{R}} |\mathcal{F}_\theta\{f\}(\xi)|^s \, d\xi\right)^{1/s} \leq r^{1/2r} s^{-1/2s} \left(\int_{\mathbb{R}} |f(t)|^r \, dt\right)^{1/r}. \tag{37}$$

Or, equivalently,

$$\left(\int_{\mathbb{R}} |\mathcal{F}_\theta\{f\}(\xi)|^s \, d\xi\right)^{1/s} \leq |\sin\theta|^{\frac{1}{s}-\frac{1}{2}} r^{1/2r} s^{-1/2s} \left(\int_{\mathbb{R}} |f(t)|^r \, dt\right)^{1/r}, \tag{38}$$

which finishes the proof of the theorem. □

Application of the above theorem will lead to the next result.

Theorem 5 (Matolcsi-Szücs Inequality). *Let* $f \in L^{r_1}(\mathbb{R}) \cap L^{r_2}(\mathbb{R})$ *such that* $1 < r_1 \leq r_2 \leq 2$, *then*

$$\|\mathcal{F}_\theta\{f\}\|_{L^{s_2}(\mathbb{R})} \leq |\operatorname{supp}(\mathcal{F}_\theta\{f\})|^{\frac{s_1-s_2}{s_1 s_2}} |\sin\theta|^{\frac{1}{s_1}-\frac{1}{2}} r_1^{1/2r_1} s_1^{-1/2s_1} |\operatorname{supp}(f)|^{\frac{r_2-r_1}{r_1 r_2}} \|f\|_{L^{r_2}(\mathbb{R})}, \tag{39}$$

where $\frac{1}{r_1} + \frac{1}{s_1} = 1$ *and* $\frac{1}{r_2} + \frac{1}{s_2} = 1$.

Proof. From Theorem 4 and Hölder's inequality we infer that

$$\begin{aligned}\|\mathcal{F}_\theta\{f\}\|_{L^{s_2}(\mathbb{R})} &\leq |\operatorname{supp}(\mathcal{F}_\theta\{f\})|^{\frac{s_1-s_2}{s_1 s_2}} \|\mathcal{F}_\theta\{f\}\|_{L^{s_1}(\mathbb{R})} \\ &\leq |\operatorname{supp}(\mathcal{F}_\theta\{f\})|^{\frac{s_1-s_2}{s_1 s_2}} |\sin\theta|^{\frac{1}{s_1}-\frac{1}{2}} r_1^{1/2r_1} s_1^{-1/2s_1} \|f\|_{L^{r_1}(\mathbb{R})}.\end{aligned} \tag{40}$$

Setting $F = \operatorname{supp}(f)$, by Hölder's inequality we obtain

$$\begin{aligned}\|f\|_{L^{r_1}(\mathbb{R})} &= \|\chi_F f\|_{L^{r_1}(\mathbb{R})} \\ &\leq \left(\int_{\mathbb{R}} |\chi_F(t)| \, dt\right)^{\frac{r_2-r_1}{r_1 r_2}} \left(\int_{\mathbb{R}} |f(t)|^{\frac{r_1 r_2}{r_1}} \, dt\right)^{\frac{r_1}{r_1 r_2}} \\ &= \left(\int_{\mathbb{R}} |\chi_F(t)| \, dt\right)^{\frac{r_2-r_1}{r_1 r_2}} \left(\int_{\mathbb{R}} |f(t)|^{r_2} \, dt\right)^{\frac{1}{r_2}} \\ &= |F|^{\frac{r_2-r_1}{r_1 r_2}} \|f\|_{L^{r_2}(\mathbb{R})} \\ &= |\operatorname{supp}(f)|^{\frac{r_2-r_1}{r_1 r_2}} \|f\|_{L^{r_2}(\mathbb{R})}.\end{aligned} \tag{41}$$

In this case χ_F is the characteristic function of F. Including (41) into (40) gives the desired result. □

Corollary 1. *Let* $f \in L^r(\mathbb{R})$ *with* $1 < r \leq 2$, *then*

$$|\sin\theta|^{\frac{1}{2}-\frac{1}{s}} r^{-1/2r} s^{1/2s} \leq |\operatorname{supp}(\mathcal{F}_\theta\{f\})|^{\frac{s-2}{2s}} |\operatorname{supp}(f)|^{\frac{2-r}{2r}}, \tag{42}$$

where $\frac{1}{r} + \frac{1}{s} = 1$.

Proof. This directly follows from the Plancherel theorem for the FRFT (18). □

5. Donoho-Stark Uncertainty Principle

As we know that Donoho-Stark uncertainty principle [39] is constructed using the basic concept of ϵ-localization of a function (signal) on a measurable set in both the space and frequency domains. Many versions of this uncertainty have been proposed for various generalized transformations (see [40,41]). Let us now remind the following definition.

Definition 4. *Given a measurable set* $T \subset \mathbb{R}$ *and let* ϵ_T *be a positive real number. It is said that a function* $f \in L^r(\mathbb{R}), 1 \leq r \leq 2$, *is* ϵ_T*-localized to* T *in* $L^r(\mathbb{R})$*-norm, if it satisfies*

$$\left(\int_{\mathbb{R}\setminus T} |f(t)|^r \, dt \right)^{1/r} = \|f - P_T f\|_{L^r(\mathbb{R})} \leq \epsilon_T \|f\|_{L^r(\mathbb{R})}, \tag{43}$$

where projection operator P_T *is given by*

$$(P_T f)(t) = (\chi_T f)(t) = \begin{cases} f(t), & t \in T \\ 0, & t \in \mathbb{R} \setminus T. \end{cases} \tag{44}$$

We also introduce a projection operator Q_T given by

$$\mathcal{F}_\theta\{Q_T f\} = \big(P_T(\mathcal{F}_\theta\{f\})\big). \tag{45}$$

Based on (43) and (45) we may define that $\mathcal{F}_\theta\{f\}$ is ϵ_T-localized to T in $L^r(\mathbb{R})$-norm if the following is satisfied:

$$\|\mathcal{F}_\theta\{f\} - \mathcal{F}_\theta\{Q_T f\}\|_{L^r(\mathbb{R})} \leq \epsilon_T \|\mathcal{F}_\theta\{f\}\|_{L^r(\mathbb{R})}. \tag{46}$$

As a direct consequence of Equation (45) we obtain the following important results.

Lemma 3. *Assume that* $|T| < \infty$. *For every* f *belongs to* $L^r(\mathbb{R})$ *with* $1 \leq r \leq 2$, *we have*

$$Q_T f(t) = \int_T \mathcal{F}_\theta\{f\}(\xi) \overline{A_\theta} e^{-i(t^2+\xi^2)\frac{\cot\theta}{2} + it\xi\csc\theta} \, d\xi, \tag{47}$$

where $|T|$ *is the Lebesgue measure on* T.

Proof. It is straightforward to show that $\mathcal{F}_\theta\{f\} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. This implies that

$$Q_T f = \mathcal{F}_\theta^{-1}\big(P_T(\mathcal{F}_\theta\{f\})\big), \tag{48}$$

which gives the desired result according to (12) and (44). □

Lemma 4. *Let* f *be in* $L^r(\mathbb{R})$. *If* $1 \leq r \leq 2$ *and* $\frac{1}{r} + \frac{1}{s} = 1$, *then*

$$\|\mathcal{F}_\theta\{Q_T f\}\|_{L^s(\mathbb{R})} \leq |\sin\theta|^{\frac{1}{s}-\frac{1}{2}} r^{1/2r} s^{-1/2s} \|f\|_{L^r(\mathbb{R})}. \tag{49}$$

Proof. An explicit computation shows that

$$\begin{aligned} \|\mathcal{F}_\theta\{Q_T f\}\|_{L^s(\mathbb{R})} &= \left(\int_T \left|\mathcal{F}_\theta\{f\}(\xi)\right|^s d\xi \right)^{1/s} \\ &\le \left(\int_{\mathbb{R}} \left|\mathcal{F}_\theta\{f\}(\xi)\right|^s d\xi \right)^{1/s} \\ &\le |\sin\theta|^{\frac{1}{s}-\frac{1}{2}} r^{1/2r} s^{-1/2s} \|f\|_{L^r(\mathbb{R})}. \end{aligned} \tag{50}$$

The proof is complete. □

Lemma 5. *Let T and E be measurable subsets of $\mathbb{R}$. For any $f \in L^s(\mathbb{R})$ with $1 \le s \le 2$ and $\frac{1}{s} + \frac{1}{r} = 1$ we have*

$$\|\mathcal{F}_\theta\{Q_T P_E f\}\|_{L^s(\mathbb{R})} \le |A_\theta| \|f\|_{L^s(\mathbb{R})} |T|^{1/r} |E|^{1/r}. \tag{51}$$

Proof. Assume that $|T| < \infty$ and $|E| < \infty$. Let us write

$$\mathcal{F}_\theta\{Q_T P_E f\} = \chi_T \mathcal{F}_\theta\{P_E f\}. \tag{52}$$

This gives

$$\|\mathcal{F}_\theta\{Q_T P_E f\}\|_{L^r(\mathbb{R})} = \left(\int_T \left|\mathcal{F}_\theta\{P_E f\}(\xi)\right|^r d\xi \right)^{1/r}. \tag{53}$$

Using Hölder's inequality we see that

$$\begin{aligned} |\mathcal{F}_\theta\{P_E f\}| &= \left|A_\theta \int_E f(t) e^{i(t^2+\xi^2)\frac{\cot\theta}{2} - it\xi\csc\theta} dt\right| \\ &\le |A_\theta| \left(\int_E |f(t)|^s dt \right)^{1/s} \left(\int_E \left|e^{i(t^2+\xi^2)\frac{\cot\theta}{2} - it\xi\csc\theta}\right|^r dt \right)^{1/r} \\ &\le |A_\theta| \left(\int_{\mathbb{R}} |f(t)|^s dt \right)^{1/s} \left(\int_{\mathbb{R}} |\chi_E|^r \right)^{1/r} \\ &= |A_\theta| \|f\|_{L^s(\mathbb{R})} |E|^{1/r}. \end{aligned} \tag{54}$$

Including (54) into (53) results in

$$\|\mathcal{F}_\theta\{Q_T P_E f\}\|_{L^r(\mathbb{R})} \le |A_\theta| \|f\|_{L^s(\mathbb{R})} |T|^{1/r} |E|^{1/r},$$

which finishes the proof of the theorem. □

Theorem 6. *Suppose that E and T are measurable subsets of $\mathbb{R}$. Suppose that $f \in L^s(\mathbb{R})$ with $1 \le s \le 2$ such that $\frac{1}{s} + \frac{1}{r} = 1$. Then for any f is ϵ_E-localized to E in $L^s(\mathbb{R})$-norm and $\mathcal{F}_\theta\{f\}$ is ϵ_T-localized to T in $L^r(\mathbb{R})$-norm we have*

$$\|\mathcal{F}_\theta\{f\}\|_{L^r(\mathbb{R})} \le \frac{\left(|A_\theta| |T|^{1/r} |E|^{1/r} + \epsilon_T |\sin\theta|^{\frac{1}{s}-\frac{1}{2}} r^{1/2r} s^{-1/2s} \right) \|f\|_{L^s(\mathbb{R})}}{1 - \epsilon_E}. \tag{55}$$

For $r = 2$ expression (55) becomes

$$|T||E||A_\theta|^2 \ge (1 - \epsilon_E - \epsilon_T |\sin\theta|)^2. \tag{56}$$

Especially when $r = 2$ *and* $\theta = \frac{\pi}{2}$*, formula* (55) *above will be reduced to*

$$|T||E| \geq 2\pi(1 - \epsilon_E - \epsilon_T)^2. \tag{57}$$

Proof. We apply (46), (49) and the triangle inequality to get

$$\begin{aligned}
&\|\mathcal{F}_\theta\{f\} - \mathcal{F}_\theta\{Q_E P_T f\}\|_{L^r(\mathbb{R})} \\
&\quad \leq \|\mathcal{F}_\theta\{f\} - \mathcal{F}_\theta\{Q_E f\}\|_{L^r(\mathbb{R})} + \|\mathcal{F}_\theta\{Q_E f\} - \mathcal{F}_\theta\{Q_E P_T f\}\|_{L^r(\mathbb{R})} \\
&\quad = \|\mathcal{F}_\theta\{f\} - \mathcal{F}_\theta\{Q_E f\}\|_{L^r(\mathbb{R})} + \|\mathcal{F}_\theta\{Q_E f - Q_E P_T f\}\|_{L^r(\mathbb{R})} \\
&\quad \leq \|\mathcal{F}_\theta\{f\} - \mathcal{F}_\theta\{Q_E f\}\|_{L^r(\mathbb{R})} + |\sin\theta|^{\frac{1}{s} - \frac{1}{2}} r^{1/2r} s^{-1/2s} \|f - P_T f\|_{L^s(\mathbb{R})} \\
&\quad \leq \epsilon_E \|\mathcal{F}_\theta\{f\}\|_{L^s(\mathbb{R})} + \epsilon_T |\sin\theta|^{\frac{1}{s} - \frac{1}{2}} r^{1/2r} s^{-1/2s} \|f\|_{L^s(\mathbb{R})}.
\end{aligned} \tag{58}$$

Applying (51) and (58) results in

$$\begin{aligned}
&\|\mathcal{F}_\theta\{f\}\|_{L^r(\mathbb{R})} \\
&\leq \|\mathcal{F}_\theta\{Q_E P_T f\|_{L^r(\mathbb{R})} + \|\mathcal{F}_\theta\{f\} - \mathcal{F}_\theta\{Q_E P_T f\}\|_{L^r(\mathbb{R})} \\
&\leq |A_\theta| \|f\|_{L^s(\mathbb{R})} |T|^{1/r} |E|^{1/r} + \epsilon_E \|\mathcal{F}_\theta\{f\}\|_{L^s(\mathbb{R})} + \epsilon_T |\sin\theta|^{\frac{1}{s} - \frac{1}{2}} r^{1/2r} s^{-1/2s} \|f\|_{L^s(\mathbb{R})} \\
&= \left(|A_\theta| |T|^{1/r} |E|^{1/r} + \epsilon_T |\sin\theta|^{\frac{1}{s} - \frac{1}{2}} r^{1/2r} s^{-1/2s} \right) \|f\|_{L^s(\mathbb{R})} + \epsilon_E \|\mathcal{F}_\theta\{f\}\|_{L^s(\mathbb{R})},
\end{aligned} \tag{59}$$

which finishes the proof of the theorem. □

Theorem 7 (FRFT Donoho-Strak'uncertainty Principle)**.** *Suppose that* E *and* T *are measurable subsets of* $\mathbb{R}$*. Suppose that* $f \in L^1(\mathbb{R}) \cap L^s(\mathbb{R})$ *with* $1 \leq s \leq 2$ *such that* $\frac{1}{s} + \frac{1}{r} = 1$*. Then for any* f *is* ϵ_E*-localized to* E *in* $L^1(\mathbb{R})$*-norm and* $\mathcal{F}_\theta\{f\}$ *is* ϵ_T*-localized to* T *in* $L^r(\mathbb{R})$*-norm we have*

$$\|\mathcal{F}_\theta\{f\}\|_{L^r(\mathbb{R})} \leq \frac{|A_\theta| |E|^{1/r} |T|^{1/r} \|f\|_{L^s(\mathbb{R})}}{(1 - \epsilon_E)(1 - \epsilon_T)}. \tag{60}$$

In particular, for $s = 2$*, Equation* (60) *becomes*

$$(1 - \epsilon_E)(1 - \epsilon_T) \leq |A_\theta| \sqrt{|E||T|}. \tag{61}$$

Proof. It follows with (46) that

$$\begin{aligned}
\|\mathcal{F}_\theta\{f\}\|_{L^r(\mathbb{R})} &\leq \|\mathcal{F}_\theta\{f\} - \mathcal{F}_\theta\{Q_T f\}\|_{L^r(\mathbb{R})} + \|\mathcal{F}_\theta\{Q_T f\}\|_{L^r(\mathbb{R})} \\
&= \|\mathcal{F}_\theta\{f\} - \mathcal{F}_\theta\{Q_T f\}\|_{L^r(\mathbb{R})} + \left(\int_T |\mathcal{F}_\theta\{f\}(\xi)|^r \, d\xi \right)^{1/r} \\
&\leq \epsilon_T \|\mathcal{F}_\theta\{f\}\|_{L^r(\mathbb{R})} + |T|^{1/r} \|\mathcal{F}_\theta\{f\}\|_{L^\infty(\mathbb{R})} \\
&\leq \epsilon_T \|\mathcal{F}_\theta\{f\}\|_{L^r(\mathbb{R})} + |T|^{1/r} |A_\theta| \|f\|_{L^1(\mathbb{R})}.
\end{aligned} \tag{62}$$

This will lead to

$$\|\mathcal{F}_\theta\{f\}\|_{L^r(\mathbb{R})} \leq \frac{|A_\theta| |T|^{1/r} \|f\|_{L^1(\mathbb{R})}}{1 - \epsilon_T}. \tag{63}$$

Now it can be immediately obtained that

$$\begin{aligned}\|f\|_{L^1(\mathbb{R})} &\leq \|f - P_E f\|_{L^1(\mathbb{R})} + \|P_E f\|_{L^1(\mathbb{R})} \\ &\leq \epsilon_E \|f\|_{L^1(\mathbb{R})} + \int_E |f(t)|\, dt \\ &\leq \epsilon_E \|f\|_{L^1(\mathbb{R})} + |E|^{1/r} \|f\|_{L^s(\mathbb{R})},\end{aligned} \tag{64}$$

which gives

$$\|f\|_{L^1(\mathbb{R})} \leq \frac{|E|^{1/r}\|f\|_{L^s(\mathbb{R})}}{1-\epsilon_E}. \tag{65}$$

Inserting (65) into (63) ends the proof of the theorem. □

Definition 5. *For $1 \leq r \leq 2$ we denote by $\mathscr{B}^r(T)$ the set of the functions $g \in L^r(\mathbb{R})$ that are bandlinited to T. It means that every $g \in \mathscr{B}^r(T)$ holds $Q_T g = g$. Moreover, it is said that f is ϵ_T-bandlimited to T in $L^r(\mathbb{R})$-norm if there exists a function $g \in \mathscr{B}^r(T)$ such that*

$$\|f - g\|_{L^r(\mathbb{R})} \leq \epsilon_T \|f\|_{L^r(\mathbb{R})}. \tag{66}$$

We present the following important results which follow immediately from the definition mentioned above.

Theorem 8. *Suppose that E and T are measurable subsets of $\mathbb{R}$. For all $g \in \mathscr{B}^r(T)$ with $1 \leq r \leq 2$ we have*

$$\|P_E g\|_{L^r(\mathbb{R})} \leq |E|^{\frac{1}{r}} |T|^{\frac{1}{r}} |A_\theta| |\sin\theta|^{\frac{1}{s}-\frac{1}{2}} r^{1/2r} s^{-1/2s} \|g\|_{L^r(\mathbb{R})}, \tag{67}$$

whenever $\frac{1}{r} + \frac{1}{s} = 1$.

Proof. From Lemma 3 and the hypothesis of the theorem we deduce that

$$g(t) = \int_T \mathcal{F}_\theta\{g\}(\xi) A_\theta e^{-i(t^2+\xi^2)\frac{\cot\theta}{2} + it\xi\csc\theta}\, d\xi. \tag{68}$$

By Hölder's inequality together with Hausdorff-Young inequality for FRFT (32) we immediately obtain

$$\begin{aligned}|g(t)| &= \left| \int_T \mathcal{F}_\theta\{g\}(\xi) A_\theta e^{-i(t^2+\xi^2)\frac{\cot\theta}{2} + it\xi\csc\theta}\, d\xi \right| \\ &\leq \int_T \left| \mathcal{F}_\theta\{g\}(\xi) A_\theta e^{-i(t^2+\xi^2)\frac{\cot\theta}{2} + it\xi\csc\theta} \right| d\xi \\ &= |T|^{\frac{1}{r}} |A_\theta| \|\mathcal{F}_\theta\{g\}\|_{L^s(\mathbb{R})} \\ &\leq |T|^{\frac{1}{r}} |A_\theta| |\sin\theta|^{\frac{1}{s}-\frac{1}{2}} r^{1/2r} s^{-1/2s} \|g\|_{L^r(\mathbb{R})}.\end{aligned} \tag{69}$$

By virtue of relation (69) above it will lead to

$$\begin{aligned}\|P_E g\|_{L^r(\mathbb{R})} &= \left(\int_E |g(t)|^r\, dt \right)^{\frac{1}{r}} \\ &\leq |E|^{\frac{1}{r}} |T|^{\frac{1}{r}} |A_\theta| |\sin\theta|^{\frac{1}{s}-\frac{1}{2}} r^{1/2r} s^{-1/2s} \|g\|_{L^r(\mathbb{R})},\end{aligned}$$

and the proof is complete. □

Theorem 9. *Let E and T be measurable subsets of* $\mathbb{R}$ *and let* $f \in L^r(\mathbb{R})$ *with* $1 \le r \le 2$ *such that* $\frac{1}{r} + \frac{1}{s} = 1$. *If f is* ϵ_E*-bandlimited to E in* $L^r(\mathbb{R})$*-norm, then the following inequality holds:*

$$\|P_E f\|_{L^r(\mathbb{R})} \le \left(|E|^{\frac{1}{r}} |T|^{\frac{1}{r}} |A_\theta| |\sin\theta|^{\frac{1}{s}-\frac{1}{2}} r^{1/2r} s^{-1/2s} (1+\epsilon_T) + \epsilon_T \right) \|f\|_{L^r(\mathbb{R})}. \tag{70}$$

Proof. Since the function f is ϵ_E-bandlimited to E in $L^r(\mathbb{R})$-norm, then there exists $g \in \mathscr{B}^r(E)$ satisfying

$$\begin{aligned} \|P_E f\|_{L^r(\mathbb{R})} &\le \|P_E g\|_{L^r(\mathbb{R})} + \|P_E(f-g)\|_{L^r(\mathbb{R})} \\ &= \|P_E g\|_{L^r(\mathbb{R})} + \left(\int_E |(f-g)(t)|^r \, dt \right)^{\frac{1}{r}} \\ &\le \|P_E g\|_{L^r(\mathbb{R})} + \left(\int_{\mathbb{R}} |(f-g)(t)|^r \, dt \right)^{\frac{1}{r}} \\ &\le \|P_E g\|_{L^r(\mathbb{R})} + \epsilon_T \|f\|_{L^r(\mathbb{R})}. \end{aligned} \tag{71}$$

It is evident that

$$\|g\|_{L^r(\mathbb{R})} \le (1+\epsilon_T)\|f\|_{L^r(\mathbb{R})}. \tag{72}$$

By inserting (67) and (72) into (71) we find

$$\begin{aligned} \|P_E f\|_{L^r(\mathbb{R})} &\le |E|^{\frac{1}{r}} |T|^{\frac{1}{r}} |A_\theta| |\sin\theta|^{\frac{1}{s}-\frac{1}{2}} r^{1/2r} s^{-1/2s} \|g\|_{L^r(\mathbb{R})} + \epsilon_T \|f\|_{L^r(\mathbb{R})} \\ &\le \left(|E|^{\frac{1}{r}} |T|^{\frac{1}{r}} |A_\theta| |\sin\theta|^{\frac{1}{s}-\frac{1}{2}} r^{1/2r} s^{-1/2s} (1+\epsilon_T) + \epsilon_T \right) \|f\|_{L^r(\mathbb{R})}, \end{aligned}$$

and the proof is complete. □

Theorem 10. *Assume the same conditions as in Theorem 9. Under the additional condition that the function f is* ϵ_E*-localized to E in* $L^r(\mathbb{R})$*-norm, then one has*

$$\frac{1-\epsilon_E-\epsilon_T}{1+\epsilon_T} \le |E|^{\frac{1}{r}} |T|^{\frac{1}{r}} |A_\theta| |\sin\theta|^{\frac{1}{s}-\frac{1}{2}} r^{1/2r} s^{-1/2s}. \tag{73}$$

Proof. With the help of Definition 4 we have

$$\begin{aligned} \|f\|_{L^r(\mathbb{R})} &\le \|f - P_E f\|_{L^r(\mathbb{R})} + \|P_E f\|_{L^r(\mathbb{R})} \\ &\le \epsilon_E \|f\|_{L^r(\mathbb{R})} + \|P_E f\|_{L^r(\mathbb{R})}. \end{aligned} \tag{74}$$

Assertion (74) is equivalent to

$$\|f\|_{L^r(\mathbb{R})} \le \frac{\|P_E f\|_{L^r(\mathbb{R})}}{1-\epsilon_E}. \tag{75}$$

Substituting (75) into (70) yields

$$\|P_E f\|_{L^r(\mathbb{R})} \le \left(\left(|E|^{\frac{1}{r}} |T|^{\frac{1}{r}} |A_\theta| |\sin\theta|^{\frac{1}{s}-\frac{1}{2}} r^{1/2r} s^{-1/2s} \right)(1+\epsilon_T) + \epsilon_T \right) \frac{\|P_E f\|_{L^r(\mathbb{R})}}{1-\epsilon_E}, \tag{76}$$

as required. □

6. Inequalities for Shannon Entropy

It is the purpose of this part to develop an analogue of Shannon entropy for the FRFT. To facilitate the narrative, we first introduce the Reńyi entropy, which is formulated in the following form:

Definition 6 ([42]). *The entropy of a probability density function ρ on $\mathbb{R}$ is expressed by*

$$\mathcal{H}_\alpha(\rho) = \frac{1}{1-\alpha} \ln\left(\int_{\mathbb{R}} [\rho(t)]^\alpha \, dt\right), \quad \alpha \geq 1. \tag{77}$$

For $\alpha \to 1$, the Reńyi entropy will lead to the Shannon entropy, that is,

$$\mathbb{E}(\rho) = -\int_{\mathbb{R}} \rho(t) \ln(\rho(t)) \, dt. \tag{78}$$

The above definition leads us to obtain the following important results.

Theorem 11. *Let $\frac{1}{\alpha} + \frac{1}{\gamma} = 2$ and $f \in L^2(\mathbb{R})$, we have*

$$2\mathcal{H}_\alpha(|f(t)|^2) + 2\mathcal{H}_\gamma\left(\left|\mathcal{F}_\theta\{f\}(\xi)\right)\right|^2\right) \geq 2\ln|\sin\theta| + \frac{1}{\gamma-1}\ln(r) + \frac{1}{\alpha-1}\ln(s), \tag{79}$$

where $\frac{1}{s} + \frac{1}{r} = 1$.

Proof. In view of (32) we get

$$\left(\int_{\mathbb{R}} \left|\mathcal{F}_\theta\{f\}(\xi)\right|^s d\xi\right)^{\frac{1}{s}} \leq |\sin\theta|^{\frac{1}{s}-\frac{1}{2}} r^{1/2r} s^{-1/2s} \left(\int_{\mathbb{R}} |f(t)|^r \, dt\right)^{\frac{1}{r}}. \tag{80}$$

Now setting $s = 2\alpha$ and $r = 2\gamma$, relation (80) is reduced to

$$\left(\int_{\mathbb{R}} \left|\mathcal{F}_\theta\{f\}(\xi)\right|^{2\alpha} d\xi\right)^{\frac{1}{2\alpha}} \leq |\sin\theta|^{\frac{\gamma-1}{2\gamma}} r^{\frac{1}{\gamma}} s^{-\frac{1}{\alpha}} \left(\int_{\mathbb{R}} |f(t)|^{2\gamma} \, dt\right)^{\frac{1}{2\gamma}}. \tag{81}$$

This means that

$$\left(\int_{\mathbb{R}} \left|\mathcal{F}_\theta\{f\}(\xi)\right|^{2\alpha} d\xi\right)^{\frac{1}{1-\alpha}} \leq |\sin\theta| r^{\frac{1}{\gamma}} s^{-\frac{1}{\alpha}} \left(\int_{\mathbb{R}} |f(t)|^{2\gamma} \, dt\right)^{\frac{1}{\gamma-1}}. \tag{82}$$

Applying $\frac{1-\alpha}{\alpha} = \frac{\gamma-1}{\gamma}$ yields

$$\left(\int_{\mathbb{R}} \left|\mathcal{F}_\theta\{f\}(\xi)\right|^{2\alpha} d\xi\right)^{\frac{1}{1-\alpha}} \leq |\sin\theta| r^{\frac{2}{\gamma-1}} s^{\frac{2}{\alpha-1}} \left(\int_{\mathbb{R}} |f(t)|^{2\gamma} \, dt\right)^{\frac{1}{\gamma-1}}. \tag{83}$$

Hence,

$$\left(\int_{\mathbb{R}} \left|\mathcal{F}_\theta\{f\}(\xi)\right|^{2\alpha} d\xi\right)^{\frac{1}{1-\alpha}} \left(\int_{\mathbb{R}} |f(t)|^{2\gamma} \, dt\right)^{\frac{1}{1-\gamma}} \geq |\sin\theta| r^{\frac{2}{\gamma-1}} s^{\frac{2}{\alpha-1}}. \tag{84}$$

Taking the logarithm of both sides of (84) above results in

$$\frac{1}{1-\alpha} \ln\left(\int_{\mathbb{R}} \left|\mathcal{F}_\theta\{f\}(\xi)\right|^{2\alpha} d\xi\right) + \frac{1}{1-\gamma} \ln\left(\int_{\mathbb{R}} |f(t)|^{2\gamma} \, dt\right) \geq \ln|\sin\theta| + \frac{2}{\gamma-1}\ln(r) + \frac{2}{\alpha-1}\ln(s). \tag{85}$$

Or, equivalently,

$$\mathcal{H}_\alpha(|f(t)|^2) + \mathcal{H}_\gamma\left(\left|\mathcal{F}_\theta\{f\}(\xi))\right|^2\right) \geq \ln|\sin\theta| + \frac{2}{\gamma-1}\ln(r) + \frac{2}{\alpha-1}\ln(s), \quad (86)$$

and the proof is complete. □

Corollary 2. *For $\alpha \to 1$ and $\gamma \to 1$ one has*

$$\mathbb{E}(|f(t)|^2) + \mathbb{E}\left(\left|\mathcal{F}_\theta\{f\}(\xi))\right|^2\right) \geq \ln|\sin\theta|, \quad (87)$$

which is a Shannon entropy related to the FRFT.

7. Conclusions

In the research study, we have introduced the definition of the fractional Fourier transform. We have provided its relation to the Fourier transform and developed this relation to explore the main properties concerning this transform. Various relations associated with the fractional Fourier transform are studied in detail. We are in the direction to apply the proposed method in image enlargement [43,44] that have many applications especially in medical image zooming and land slides detection.

Author Contributions: Conceptualization, M.B.; Formal analysis, S.A.A.K.; Funding acquisition, S.A.A.K.; Investigation, M.B. and S.A.A.K.; Methodology, M.B. and S.A.A.K.; Resources, S.A.A.K.; Validation, M.B. and S.A.A.K.; Writing—original draft, M.B.; Writing—review & editing, M.B. and S.A.A.K. All authors have read and agreed to the published version of the manuscript.

Funding: Research Management Centre, Universiti Malaysia Sabah, through the UMS/PPI-DPJ1 Journal Article Fund.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The second author was fully supported by Universiti Malaysia Sabah in Malaysia. The authors give special thanks to the Faculty of Computing and Informatics at Universiti Malaysia Sabah for the computing facilities support. The authors are thankful to the anonymous reviewers for careful reading of manuscript, comments and suggestions for the improvement of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

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